Norwegian University of Science and Technology
Institutt for matematiske fag

MA0002 Brukerkurs i Matematikk B
Vår 2024

Solutions to exercise set 7

1 Use integration by parts two times to solve the integral

$$\int e^x \sin(x)dx$$

We chose $u_1(x) = e^x$ and $v_1'(x) = \sin x$ such that $u_1'(x) = e^x$ and $v_1(x) = -\cos x$. Then we get

$$\int e^x \sin(x)dx = C - e^x \cos x - \int(-e^x \cos(x))dx = C - e^x \cos x + \int e^x \cos(x)dx$$

Then we chose vi $u_2(x) = e^x$ and $v_2'(x) = \cos x$ such that $u_2'(x) = e^x$ and $v_2(x) = \sin x$. Then we get

$$C - e^x \cos x + \int e^x \cos(x)dx = C - e^x \cos x + e^x \sin x - \int e^x \sin(x)dx$$

Then we can get that

$$\begin{aligned}
\int e^x \sin(x)dx &= C - e^x \cos x + e^x \sin x - \int e^x \sin(x)dx \\
2\int e^x \sin(x)dx &= C - e^x \cos x + e^x \sin x \\
\int e^x \sin(x)dx &= C - \frac{1}{2}e^x \cos x + \frac{1}{2}e^x \sin x
\end{aligned}$$

And we are done.

2 Use partial fraction decomposition to solve the integral

$$\int_2^3 \frac{x+1}{x(x-1)}dx$$

We start by letting A and B be such that

$$\frac{x+1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

For this to hold

$$x + 1 = Ax - A + Bx$$

So we get the linear system

$$\begin{aligned}
A + B &= 1 \\
-A &= 1
\end{aligned}$$

which has the solution $A = -1, B = 2$. So we get

$$\begin{aligned}
\int_2^3 \frac{x+1}{x(x-1)}dx &= \int_2^3 \left(-\frac{1}{x} + \frac{2}{x-1}\right) dx \\
&= [-\ln|x| + 2\ln|x-1|]_2^3 \\
&= (-\ln 3 + 2\ln 2) - (-\ln 2 + 2\ln 1) \\
&= 3\ln 2 - \ln 3 \approx 0.9808
\end{aligned}$$

$\boxed{3}$ Solve the differential equation with initial values

$$\frac{dy}{dx} = ye^x, \text{ with } y_0 = 1, x_0 = 1$$

We start by rearranging the equation to

$$\frac{dy}{y} = e^x dx$$

we then integrate on both sides and compute

$$\begin{aligned}
\ln|y| &= e^x + C \\
|y| &= e^{e^x + C} \\
y &= Ce^{e^x}
\end{aligned}$$

Then we evaluete with $y_0 = 1$ and $x_0 = 1$ and get that $C = (e^e)^{-1} = e^{-e}$ so we have the solution

$$y = e^{e^x - e}$$

$\boxed{4}$ Let

$$\frac{dy}{dx} = (y-1)(y+1)$$

Find the points of equilibia and find out if they are stable or unstable.

First we let $g(y) = (y-1)(y+1)$ and see that the points $y_1 = -1$ and $y_2 = 1$ are solutions for $g(y) = 0$, which means that these are the points of equilibria. Then we compute $g'(y) = 2y$. And compute $g'(y_2) = 2 > 0$ and $g'(y_1) = -2 < 0$, which means that y_1 is a stable point of equilibria and y_2 is an unstable point of equilibria.

$\boxed{5}$ Find the augmented matrix and use it to solve the linear system

$$\begin{aligned}
x - y &= 1 \\
2x + y &= -1
\end{aligned}$$

We set up the augmented matrix and compute

$$\begin{aligned}
\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & -1 \end{array}\right] \; -2 \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -3 \end{array}\right] \; \cdot\left(\tfrac{1}{3}\right) \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -1 \end{array}\right] \; + \\
\sim \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -1 \end{array}\right]
\end{aligned}$$

So we have the solution $x = 0, y = -1$.

$\boxed{6}$ In this task we will consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 7 & 7 & 7 \\ 6 & 6 & 6 \end{bmatrix}, C = \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ -1 & 3 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

a) Compute $A + B$.

$$A + B = \begin{bmatrix} 1+7 & 2+7 & 3+7 \\ 2+6 & 3+6 & 4+6 \end{bmatrix} = \begin{bmatrix} 8 & 9 & 10 \\ 8 & 9 & 10 \end{bmatrix}$$

b) Compute $A \cdot C$ and $C \cdot A$.

$$A \cdot C = \begin{bmatrix} 7 & 14 \\ 12 & 21 \end{bmatrix}$$

$$C \cdot A = \begin{bmatrix} 8 & 13 & 18 \\ 6 & 11 & 16 \\ 5 & 7 & 9 \end{bmatrix}$$

c) If it exists, compute the inverse matrix of D.

We set up the matrix for computing the inverse, and compute it

$$\begin{aligned}
&\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} -1 \quad 1 \\ \leftarrow \\ \leftarrow \end{array} \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 3 & 4 & 1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \\ \updownarrow \\ \end{array} \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \\ \cdot(\frac{1}{3}) \\ \end{array} \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \leftarrow \\ \leftarrow \\ -\frac{4}{3} \quad -3 \end{array} \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & -3 & 0 \\ 0 & 1 & 0 & 5/3 & -4/3 & 1/3 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} \leftarrow \\ -2 \\ \end{array} \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & -1/3 & -2/3 \\ 0 & 1 & 0 & 5/3 & -4/3 & 1/3 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]
\end{aligned}$$

Then we have that

$$D^{-1} = \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ 5/3 & -4/3 & 1/3 \\ -1 & 1 & 0 \end{bmatrix}$$