Norwegian University of Science and Technology
Department of Mathematical Sciences

MA0002 Mathematical Methods B
Spring 2024

Exercise set 8

$\boxed{1}$ In this exercise we look at what a Leslie matrix tells us about. Let

$$L = \begin{pmatrix} 0 & 5 & 2 \\ 0.8 & 0 & 0 \\ 0 & 0.3 & 0 \end{pmatrix}$$

be a Leslie matrix describing a population.

(a) Determine the average number of female offsprings of the two-year-olds.

(b) Determine the average number of female offsprings of the one-year-olds.

(b) What can you say about the rate of survival of zero-year-olds and one-year-olds in this population?

Solution.

(a) The two-year old females have an average of 2 female offspring in this population.

(b) The one-year old females have an average of 5 female offspring in this population.

(c) From the Leslie matrix we see that 80% of the zero-year-olds and 30% of the one-year-olds survive until the next breading season.

$\boxed{2}$ Assume that we have a population consisting of zero-year-olds, one-year-olds and two-year-olds. Assume that 80% of the female zero-year-olds and 10% of the female one-year-olds survive until the next breeding season. Assume further that female one-year-olds have an average of 1.6 female offspring and that female two-year-olds have an average of 3.9 female offspring. We assume that at time $t = 0$ the population consists of 1000 female zero-year-olds, 100 female one-year-olds and 20 female two-year-olds.

(a) Find the Leslie matrix L that describes the situation above.

(b) Find the age distribution at time $t = 2$. That means; how many female zero-year-olds, one-year-olds and two-year-olds does the population consist of after two years?

Solution
(a). We can describe the situation with the following Leslie matrix

$$L = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix}.$$

(b) We know that $N(t+1) = LN(t)$. We start by finding $N(1)$.

$$N(1) = LN(0) = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 100 \\ 20 \end{pmatrix} = \begin{pmatrix} 238 \\ 800 \\ 10 \end{pmatrix}.$$

We can now calculate $N(2)$.

$$N(2) = LN(1) = \begin{pmatrix} 0 & 1.6 & 3.9 \\ 0.8 & 0 & 0 \\ 0 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} 238 \\ 800 \\ 10 \end{pmatrix} = \begin{pmatrix} 1319 \\ 190 \\ 80 \end{pmatrix}.$$

This means that after 2 years, the population will consist of 1319 female zero-year-olds, 190 female one-year-olds and 80 female two-year-olds.

$\boxed{3}$ In this exercise we will consider the vectors

$$x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad y = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \text{og} \quad z = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

(a) Calculate $2x - y$ and illustrate the calculation with a drawing in the plane.

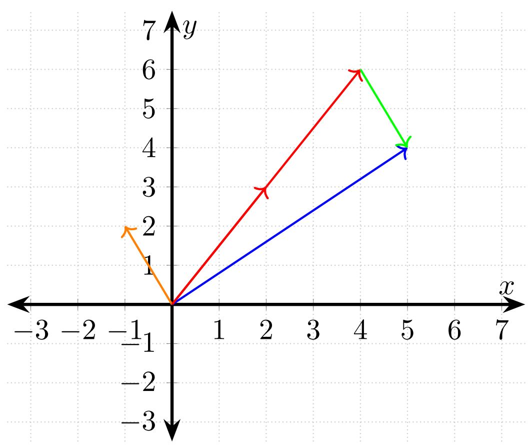
(b) Find the polar coordinates of the vector z and draw the vector.

Solution.

(a) We see that

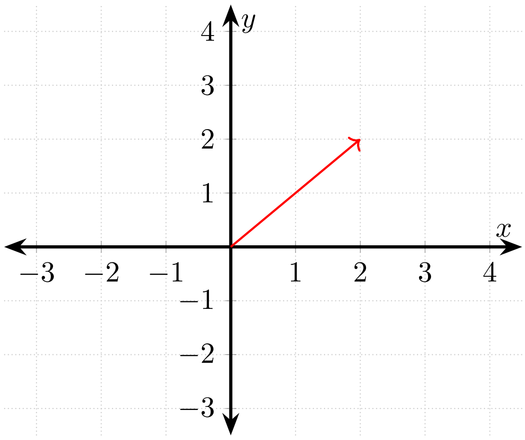
$$2x - y = 2\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Exercise 3a



(b) We see that $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and that $\theta = \arctan 2/2 = \frac{\pi}{4}$.

Exercise 3b



4 Given the following vectors in polar coordinates (r, θ), find the cartesian coordinates of the vector.

(a) $r = 4$, $\theta = \frac{\pi}{2}$

(b) $r = 2$, $\theta = \frac{4\pi}{3}$

Solution.

(a) We see that $x_1 = r\cos\theta = 4\cos\frac{\pi}{2} = 4 * 0 = 0$ and that $x_2 = r\sin\theta = 4\sin\frac{\pi}{2} = 4$. Thus, in cartesian coordinates we have

$$x = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

(b) We see that $x_1 = r\cos\theta = 2\cos\frac{4\pi}{3} = -1$, and that $x_2 = r\sin\theta = 2\sin\frac{4\pi}{3} = -\sqrt{3}$. Thus it follows that in cartesian coordinates we have

$$x = \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}.$$

5 For each of the maps, determine whether they are linear or not.

(a)

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix}$$

(b)

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + 2 \\ x_2 \end{pmatrix}$$

Solution.

(a) We see that T is linear since

$$T\left(\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}\right) = \begin{pmatrix} 2(x_1+y_1) \\ 3(x_2+y_2) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 3x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 \\ 3y_2 \end{pmatrix} = T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) + T\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right).$$

(b) We see that T is not linear since $f(x) = 3x+2$ is not linear. We can see this by calculating $f(1+1) = f(2) = 8 \neq 10 = f(1) + f(1)$.

$\boxed{6}$ Let

$$A = \begin{pmatrix} 3 & 6 \\ 1 & 4 \end{pmatrix} \quad \text{and,} \quad B = \begin{pmatrix} -1 & 4 \\ 0 & -2 \end{pmatrix}.$$

For each of the matrices A and B, find the eigenvalues and the corresponding eigenvectors.

Solution

(a) To find the eigenvalues we must solve the equation $\det(A - \lambda I) = 0$. We see that

$$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 6 \\ 1 & 4-\lambda \end{pmatrix} = \lambda^2 - 7\lambda - 6 = (\lambda-1)(\lambda-6).$$

Thus, it follows that we have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 6$. To find the corresponding eigenvectors, we must solve the equation $Ax = \lambda x$, which is equivalent to solving $(A-\lambda I)x = 0$. That is, we must solve

$$\begin{aligned} 2x_1 + 6x_2 &= 0 \\ x_1 + 3x_2 &= 0. \end{aligned}$$

, to find the eigenvectors corresponding to $\lambda_1 = 1$. We see that this is equivalent with $x_1+3x_2 = 0$, which has the solution $x_1 = -3x_2$. Thus the eigenvectors corresponding to $\lambda_1$is all vectors on the form

$$v \in \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix} y : y \in \mathbb{R} \right\}.$$

Letting $y = 1$ we see that $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an example of an eigenvector.

We now find the eigenvectors corresponding to $\lambda_2 = 6$ by solving

$$\begin{aligned} (3-6)x_1 + 6x_2 &= 0 \\ x_1 + (4-6)x_2 &= 0, \end{aligned}$$

which is equivalent to $x_1 = 2x_2$. Thus, the eigenvectors belonging to λ_1 is all vectors

$$v \in \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} y : y \in \mathbb{R} \right\}.$$

By letting $y = 1$ we see that an example of an eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(b) We solve the equation $\det(B - \lambda I) = 0$ to find the eigenvalues. We see that

$$\det(B - \lambda I) = \det \begin{pmatrix} -1-\lambda & 4 \\ 0 & -2-\lambda \end{pmatrix} = -(1+\lambda) * -(2+\lambda) - 4 * 0 = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1),$$

so we have the two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$. We find the eigenvectors beloning to λ_1 by solving $B + 2I = 0$, that is

$$\begin{aligned} x_1 + 4x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus, the eigenvectors are all the vectors $\begin{pmatrix} -4 \\ 1 \end{pmatrix} y$, where $y \in \mathbb{R}$. To find the eigenvectors beloning to $\lambda_2 = -1$ we solve $B + I = 0$,that is,

$$\begin{aligned} 4x_2 &= 0 \\ -x_2 &= 0 \end{aligned}$$

Thus the eigenvectors belonging to the eigenvalue $\lambda_2 = -1$ is all vectors on the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix} y$, where $y \in \mathbb{R}$.