Norwegian University of Science and Technology
Department of Mathematical Sciences

MA0002 Mathematical Methods B
Spring 2023

Exercise set 9

$\boxed{1}$ Let

$$L = \begin{bmatrix} 1 & 3 \\ 0.7 & 0 \end{bmatrix}$$

be the Leslie matrix for a population with two age groups.

a) Determine both eigenvalues.
To find the eidenvalues, we solve the polynomial.

$$\begin{aligned} 0 &= \det(L - \lambda I) = (1-\lambda)(-\lambda) - 3 \cdot 0.7 \\ 0 &= \lambda^2 - \lambda - 2.1 \end{aligned}$$

From this polynomial we get the solutions $\lambda_1 \approx 2.03$ and $\lambda_2 \approx -1.03$, which are the two eigenvalues of L.

b) Give a biological interpretation of the largest of the two eigenvalues.
The largest of the eigenvalues, λ_1, describes the growth rate of the population.

c) Find the stable age distribution.
To find the stable growth rate, we need to find the eigenvector of λ_1. We do this by finding the vector which satisfy

$$\begin{bmatrix} 1 & 3 \\ 0.7 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is the same as a solution of

$$\begin{aligned} x_1 + 3x_2 &= \lambda_1 x_1 \\ 0.7x_1 &= \lambda_1 x_2 \end{aligned}$$

which can be simplefyed to

$$\begin{aligned} x_1 + \frac{3}{(1-\lambda_1)} x_2 &= 0 \\ x_1 - \frac{\lambda_1}{0.7} x_2 &= 0 \end{aligned}$$

which is

$$\begin{aligned} x_1 - 2.9x_2 &= 0 \\ x_1 - 2.9x_2 &= 0 \end{aligned}$$

so

$$x_1 = 2.9x_2$$
$$10x_1 = 29x_2$$

So the vector

$$\mathbf{x} = \begin{bmatrix} 10 \\ 29 \end{bmatrix}$$

is a stable age distrebution.

$\boxed{2}$ Let $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

a) Find $\mathbf{x} - \mathbf{y}$.

$$\mathbf{x} - \mathbf{y} = \begin{bmatrix} 2-1 \\ -1-(-3) \\ 0-2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

b) Find $4\mathbf{x} + 2\mathbf{y}$.

$$4\mathbf{x} + 2\mathbf{y} = \begin{bmatrix} 10 \\ -10 \\ 4 \end{bmatrix}$$

c) Find $-\mathbf{x} + 2\mathbf{y}$.

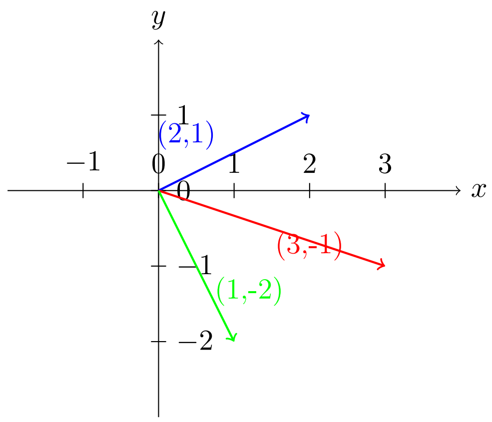
$$-\mathbf{x} + 2\mathbf{y} = \begin{bmatrix} 0 \\ -5 \\ 4 \end{bmatrix}$$

d) Find the length of $\mathbf{x}$.

$$||\mathbf{x}|| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.$$

$\boxed{3}$ Let $A = (2, 1)$ and $B = (3, -1)$. Find the vector representation of $B - A$ (same as $\overrightarrow{AB}$). Then sketch A, B and $B - A$.

$$B - A = \begin{bmatrix} 3-2 \\ -1-1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$



$\boxed{4}$ Find the dot product of $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$.

$$\mathbf{x} \cdot \mathbf{y} = 1 \cdot 3 + 0 \cdot 5 + -1 \cdot (-2) = 5.$$

$\boxed{5}$ Let $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. Find a $\mathbf{y}$ such that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

We can chose any vector $\mathbf{y}$ such that the dot product of $\mathbf{x}$ and $\mathbf{y}$ is zero. One example of this is the vector

$$\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$\boxed{6}$ A triangle has vertices at coordinates $P = (0,0), Q = (2,0)$ and $R = (1,1)$.

a) Find the lengths of all three sides.

$$\begin{aligned} ||P - Q|| &= \sqrt{(-2)^2 + 0^2} = 2 \\ ||Q - R|| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ ||R - P|| &= \sqrt{1^2 + 1^2} = \sqrt{2} \end{aligned}$$

b) Use the dot product to find all three angles.

$$\begin{aligned} (Q - P) \cdot (Q - R) &= (2,0) \cdot (1,-1) = 2 \\ (P - Q) \cdot (P - R) &= (-2,0) \cdot (-1,-1) = 2 \\ (R - P) \cdot (R - Q) &= (1,1) \cdot (-1,1) = 0 \end{aligned}$$

Then we set the angles between QP and QR, between PQ and PR and between RP and RQ as θ_1, θ_2 and θ_3, we get that

$$\begin{aligned} \cos\theta_1 &= \frac{(Q - P) \cdot (Q - R)}{||P - Q|| \cdot ||Q - R||} = \frac{2}{2 \cdot \sqrt{2}} \\ \cos\theta_2 &= \frac{(P - Q) \cdot (P - R)}{||P - Q|| \cdot ||R - P||} = \frac{2}{2 \cdot \sqrt{2}} \\ \cos\theta_3 &= \frac{(R - P) \cdot (R - Q)}{||Q - R|| \cdot ||R - P||} = 0 \end{aligned}$$

Which means that $\theta_1 = 45^o$, $\theta_2 = 45^o$ and $\theta_3 = 0^o$.

$\boxed{7}$ Find a representation of the line through the two points $(2,1)$ and $(1,0)$.

We find a representation of the form $y = ax + b$.

$$a = \frac{1 - 0}{2 - 1} = 1$$

So we have $y = x + b$. To find b we sett in for the point $(1, 0)$

$$0 = 1 + b$$

So $b = -1$ and we have the representation

$$y = x - 1.$$