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Solutions to exercise set 12

$\boxed{1}$ Let $f(x_1, x_2) = x_1 e^{x_2}$, where $x_1(t) = e^t$ and $x_2(t) = t^2$. Find the derivative of $w = f(x, y)$ with respect to t when $t = 0$, that is find $\frac{df}{dt}\big|_{t=0}$.

Solution.
We apply the chain rule:

$$\frac{df}{dt} = \frac{df}{dx_1}\frac{dx_1}{dt} + \frac{df}{dx_2}\frac{dx_2}{dt} = e^{x_2(t)}e^t + x_1(t)e^{x_2(t)}2t.$$

Evaluating in $t = 0$ we see that

$$\frac{df}{dt}\Big|_{t=0} = e^{x_2(0)}e^0 + x_1(+)e^{x_2(0)} \cdot 0 = 1.$$

$\boxed{2}$ Let $f(x_1, x_2) = \ln(x_1 x_2 - x_1^2)$ where $x_1(t) = t^2$ and $x_2(t) = t$. Find the derivative of $w = f(x_1, x_2)$ with respect to t when $t = 5$, that is find $\frac{df}{dt}\big|_{t=5}$.

Solution.
We apply the chain rule:

$$\frac{df}{dt} = \frac{df}{dx_1}\frac{dx_1}{dt} + \frac{df}{dx_2}\frac{dx_2}{dt} = \frac{x_2 - 2x_1}{x_1 x_2 - x_1^2}2t + \frac{x_1}{x_1 x_2 - x_1^2}.$$

We have $x_1(5) = 25$ and $x_2(5) = 5$, thus evaluating in $t = 5$ we see that

$$\frac{df}{dt}\Big|_{t=5} = \frac{5 - 2 \cdot 25}{5 \cdot 25 - 25^2} \cdot 10 + \frac{25}{5 \cdot 25 - 25^2} = \frac{17}{20}.$$

$\boxed{3}$ Find the gradient of the following functions.

(a)
$$f(x_1, x_2) = x_1(x_1^2 - x_2^2)^{2/3}$$

(b)
$$g(x_1, x_2) = \sin(3x_1^2 - 2x_2)$$

Solution(a).
We know that the gradient is given by

$$\nabla f(x_1,x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} f(x_1,x_2) \\ \\ \frac{\partial f}{\partial x_2} f(x_1,x_2) \end{pmatrix}.$$

We calculate the partial derivatives by using the product rule and the chain rule:

$$\frac{\partial f}{\partial x_1}(x_1,x_2) = \left(x_1^2 - x_2^2\right)^{\frac{2}{3}} + \frac{4\cdot x_1^2}{3\cdot (x_1^2 - x_2^2)^{\frac{1}{3}}} = \frac{7\cdot x_1^2 - 3\cdot x_2^2}{3\cdot (x_1^2 - x_2^2)^{\frac{1}{3}}}$$

and

$$\frac{\partial f}{\partial x_2}(x_1,x_2) = \frac{-4\cdot x_1 \cdot x_2}{3\cdot (x_1^2 - x_2^2)^{\frac{1}{3}}},$$

so the gradient ∇f is given by

$$\nabla f(x_1,x_2) = \frac{1}{3\cdot (x_1^2 - x_2^2)^{\frac{1}{3}}} \cdot \begin{pmatrix} 7\cdot x_1^2 - 3\cdot x_2^2 \\ -4\cdot x_1\cdot x_2 \end{pmatrix}.$$

Solution(b).
We know that the gradient is given by

$$\nabla g(x_1,x_2) = \begin{pmatrix} \frac{\partial g}{\partial x_1} g(x_1,x_2) \\ \\ \frac{\partial g}{\partial x_2} g(x_1,x_2) \end{pmatrix}.$$

We calculate the partial derivatives by applying the chain rule and see that

$$\frac{\partial g}{\partial x_1}(x_1,x_2) = 6x_1 \cos(3x_1^2 - 2x_2),$$

and

$$\frac{\partial g}{\partial x_2}(x_1,x_2) = -2\cos(3x_1^2 - 2x_2).$$

We thus obtain

$$\nabla g(x_1,x_2) = \begin{pmatrix} 6x_1\cos(3x_1^2 - 2x_2) \\ \\ -2\cos(3x_1^2 - 2x_2) \end{pmatrix} = \cos(3x_1^2 - 2x_2)\begin{pmatrix} 6x_1 \\ -2 \end{pmatrix}.$$

$\boxed{4}$ Let $f(x_1,x_2) = x_1^3 x_2^2$. Find the directional derivative of $f(x_1,x_2)$ in the point $(2,3)$ and the direction $(-2,1)^T$.

Solution. We know that the directional derivative of the function $f(x_1,x_2)$ in the point (x_0,y_0) in the direction of the unit vector $\mathbf{u} = (u_1,u_2)^T$ is given by

$$D_{\mathbf{u}} f(x_0,y_0) = (\nabla f(x_0,y_0))\cdot \mathbf{u}.$$

We start by finding the gradient of the function $f(x_1, x_2)$.

$$\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} f(x_1, x_2) \\ \\ \frac{\partial f}{\partial x_2} f(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 3x_1^2 x_2^2 \\ \\ 2x_1^3 x_2 \end{pmatrix}.$$

We evaluate at the point$(2, 3)$ and see that

$$\nabla f(2, 3) = \begin{pmatrix} 108 \\ 48 \end{pmatrix}.$$

We normalize the directional vector $(-2, 1)^T$ and see that

$$\mathbf{u} = \frac{1}{\sqrt{(-2)^2 + 1^2}} \cdot (-2, 1)^T = \frac{1}{\sqrt{5}}(-2, 1)^T.$$

Thus we now know that the directional derivative is given by

$$D_{\mathbf{u}} f(x_0, y_0) = (\nabla f(x_0, y_0)) \cdot \mathbf{u} = \begin{pmatrix} 108 \\ 48 \end{pmatrix} \cdot \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \frac{-168\sqrt{5}}{5}.$$

$\boxed{5}$ Let $f(x_1, x_2) = e^{x_1} \cos x_2$. In which direction does the function $f(x_1, x_2)$ increase the most in the point $(0, \pi/2)$?

Solution.
We know that $f(x_1, x_2)$ increases the most in the point (a, b) in the direction of $\nabla f(a, b)$. We start by finding the gradient:

$$\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} f(x_1, x_2) \\ \\ \frac{\partial f}{\partial x_2} f(x_1, x_2) \end{pmatrix} = \begin{pmatrix} e^{x_1} \cos(x_2) \\ \\ -e^{x_1} \sin(x_2) \end{pmatrix}.$$

Evaluating in $(0, \pi/2)$ we see that $\nabla f(0, \pi/2) = (0, 1)^T$. Thus we know that the function $f(x_1, x_2)$ increases the fastest in the direction $(0, 1)^T$ in the point $(0, \pi/2)$.

$\boxed{6}$ Given the following function; find all the critical points and classify them.

(a)

$$f(x_1, x_2) = -2x_1^2 - x_2^2 + 3x_2$$

(b)

$$f(x_1, x_2) = x_1^3 - 2x_2^2 + 2x_1 x_2$$

Solution.

(a) To find the gradient of the function f we start by calculating the first order partial derivatives.

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = -4x_1,$$

and

$$\frac{\partial f(x_1,x_2)}{\partial x_2} = 2x_2 + 3$$

Thus we know that

$$\nabla f(x_1,x_2) = \begin{pmatrix} -4x_1 \\ 2x_2 + 3 \end{pmatrix}.$$

We find the critical points of the function f by solving the equation $\nabla f(x_1,x_2) = (0,0)^T$ and see that the only critical point of the function f is the point $(0,-3/2)$. To classify the critical point we find the second order partial derivatives.

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = -4$$
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} = 0$$
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} = 0$$
$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = 2$$

Thus the Hessian matrix is given by

$$\mathrm{Hess} f(x,y) = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}$$

In particular we see that the determinant is $D = -8$ for all (x_1,x_2) so the point $(0,-3/2)$ is a saddle point. This means that f does not have any local maxima or minima.

(b) To find the gradient of the function f we start by calculating the first order partial derivatives.

$$\frac{\partial f(x_1,x_2)}{\partial x_1} = 3x_1^2 + 2x_2,$$

and

$$\frac{\partial f(x_1,x_2)}{\partial x_2} = -4x_2 + 2x_1.$$

Thus it follows that

$$\nabla f(x_1,x_2) = \begin{pmatrix} 3x_1^2 + 2x_2 \\ -4x_2 + 2x_1 \end{pmatrix}.$$

We find the critical points by solving the equation $\nabla f(x_1,x_2) = 0$, that is

$$3x_1^2 + 2x_2 = 0 \tag{1}$$
$$-4x_2 + 2x_1 = 0. \tag{2}$$

We see that the second inequality is equivalent with the equation $x_1 = 2x_2$. By inserting this in the first equation we obtain the equation $3 \cdot (2y)^2 + 2y = 0$. Which has the solutions $y = 0$ and $y = -1/6$. By then solving for x we see that f has the two critical points $(0, 0)$ and $(-1/3, -1/6)$. To classify the critical points we need to find the second order partial derivatives.

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 6x_1$$
$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = 2$$
$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = 2$$
$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = -4$$

Thus the Hessian matrix is given by

$$\mathrm{Hess} f(x, y) = \begin{pmatrix} 6x_1 & 2 \\ 2 & -4 \end{pmatrix}$$

In particular we see that

$$\det(\mathrm{Hess} f(0, 0)) = -4 < 0,$$

so $(0, 0)$ is a saddle point. Further we see that

$$\det(\mathrm{Hess} f(-1/3, -1/6)) = 8 - 4 = 4 > 0,$$

and that

$$\mathrm{tr}(\mathrm{Hess} f(-1/3, -1/6) = -6 < 0,$$

this means that $(-1/3, -1/6)$ is a local maximum.