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1 Evaluate the following integrals:

a.
$$\int \frac{2x}{x+4}\,\mathrm{d}x$$

b.
$$\int \frac{x^2+2x+3}{x-1}\,\mathrm{d}x$$

c.
$$\int \frac{2x^3-x^2+2x-2}{x^2+2}\,\mathrm{d}x$$

Solution exercise 1a.

$$\int \frac{2x}{x+4}\,\mathrm{d}x = \int \frac{2x+8-8}{x+4}\,\mathrm{d}x = \int 2 + \frac{-8}{x+4}\,\mathrm{d}x = 2x - 8\ln|x+4| + C$$

Solution exercise 1b.

$$\begin{aligned}\int \frac{x^2+2x+3}{x-1}\,\mathrm{d}x &= \int \frac{x^2-2x+1+4x-4+6}{x-1}\,\mathrm{d}x\\ &= \int \frac{(x-1)^2+4(x-1)+6}{x-1}\,\mathrm{d}x\\ &= \int x-1+4+\frac{6}{x-1}\,\mathrm{d}x\\ &= \frac{1}{2}x^2+3x+6\ln|x-1|+C\end{aligned}$$

Solution exercise 1c.

$$\begin{aligned}\int \frac{2x^3-x^2+2x-2}{x^2+2}\,\mathrm{d}x &= \int \frac{(x^2+2)(2x-1)-2x}{x^2+2}\,\mathrm{d}x\\ &= \int 2x-1-\frac{2x}{x^2+2}\,\mathrm{d}x\\ &= x^2-x-\log(x^2+2)+C\end{aligned}$$

Note: In these exercises we have chosen to rewrite the nominator in a suitable way to easy the calculations, if you prefer you could instead use polynomial division.

$\boxed{2}$ Find the partial fraction decomposition of the following integrals. When you have found an answer; check that your solution is correct by adding all the parts of the partial fraction decomposition together again.

a.
$$f(x) = -\frac{4x-3}{(x-2)(2x+1)}$$

b.
$$g(x) = \frac{x-10}{(3x-2)(2x+1)}$$

c.
$$h(x) = \frac{1}{(x+1)(x-1)^2}$$

Solution exercise 2a.
Since the denominator is a product of two linear factors $(x-a)(x-b)$ with $a \neq b$ we know that there exist constants A and B such that

$$-\frac{4x-3}{(2x+1)(x-2)} = \frac{A}{2x+1} + \frac{B}{x-2}.$$

To find the constants A and B we write sum the right hand side of the equation above so we can write it as one fraction;

$$\frac{A}{2x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(2x-1)}{(2x+1)(x-2)} = \frac{(2B+A)x-2A+B}{(2x+1)(x-2)}$$

Thus it follows that $2B+A=-4$ and $-2A+B=3$. Solving these equations we see that $A=-2$ and $B=-1$. Then it follows that

$$f(x) = -\frac{4x-3}{(2x+1)(x-2)} = -\frac{2}{2x+1} - \frac{1}{x-2}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$-\frac{2}{2x+1} - \frac{1}{x-2} = \frac{-2(x-2)-(2x+1)}{(2x+1)(x-2)} = \frac{-4x+3}{(2x+1)(x-2)}.$$

Solution exercise 2b.
We do the same as in the previous exercise and calculate:

$$\frac{x-10}{(3x-2)(2x+1)} = \frac{A}{3x-2} + \frac{B}{2x+1} = \frac{(2x+1)A+(3x-2)B}{(3x-2)(2x+1)} = \frac{x(2A+3B)+A-2B}{(3x-2)(2x+1)}$$

To find the constants A and B we must solve the equations $2A+3B=1$ and $A-2B=-10$. It follows that $A=-4$ and $B=3$. We can now see that

$$\frac{x-10}{(3x-2)(2x+1)} = \frac{-4}{3x-2} + \frac{3}{2x+1}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$\frac{-4}{3x-2} + \frac{3}{2x+1} = \frac{-4(2x+1)+3(3x-2)}{(3x-2)(2x+1)} = \frac{x-10}{(3x-2)(2x+1)}.$$

Solution exercise 2c.
We observe that the denumerator consits of repiting linear factors and we must thus use a slightly different approach than in the previous exercises. We know there exists constants A, B and C such that

$$\frac{1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

To find the constants A, B and C we calculate the right hand side:

$$\begin{aligned} \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} &= \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{x(x+1)^2} \\ &= \frac{(A+B)x^2 + (-2A+C)x + A - B + C}{(x+1)(x-1)^2}. \end{aligned}$$

We see that it must be the case that $A+B=0$, $-2A+C=0$ and $A-B+C=1$. Solving this system of equations we see that $A=\frac{1}{4}$, $B=-\frac{1}{4}$ and $C=\frac{1}{2}$. Thus it follows that

$$\frac{1}{(x+1)(x-1)^2} = \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}.$$

To check that we have not made any mistakes, we sum the fractions and see that we get the expresion we started with:

$$\begin{aligned} \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} &= \frac{(x-1)^2 - (x+1)(x-1) + 2(x+1)}{4(x+1)(x-1)^2} \\ &= \frac{x^2-2x+1-x^2+1+2x+2}{4(x+1)(x-1)^2} = \frac{1}{(x+1)(x-1)^2}. \end{aligned}$$

$\boxed{3}$ Evaluate the following integrals by using the partial fraction decompositions obtained in the previous exercise

a.
$$\int -\frac{4x-3}{(x-2)(2x+1)}\,\mathrm{d}x$$

b.
$$\int \frac{x-10}{(3x-2)(2x+1)}\,\mathrm{d}x$$

c.
$$\int \frac{1}{(x+1)(x-1)^2}\,\mathrm{d}x$$

Solution exercise 3a.

$$\int -\frac{4x-3}{(x-2)(2x+1)}\,\mathrm{d}x = \int -\frac{2}{2x+1} - \frac{1}{x-2}\,\mathrm{d}x = -\ln|2x+1| - \ln|x-2| + C$$

To calculate the integrals above you could use the substitution $u = 2x+1$ and $u = x-1$.
Solution exercise 3b.

$$\int \frac{x-10}{(3x-2)(2x+1)}\,\mathrm{d}x = \int \frac{-4}{3x-2} + \frac{3}{2x+1} = -\frac{4}{3}\ln|3x-2| + \frac{3}{2}\ln|2x+1| + C$$

Solution exercise 3c.

$$\begin{aligned}\int \frac{1}{(x+1)(x-1)^2}\,\mathrm{d}x &= \int \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}\,\mathrm{d}x \\ &= \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln|x-1| - \frac{1}{2(x-1)} + C\end{aligned}$$

$\boxed{4}$ Evaluate the integrals:

a.
$$\int \frac{2x^2+5x+11}{(x+2)^2(x-1)}\,\mathrm{d}x$$

b.
$$\int \frac{1}{(x-2)(x+1)}\,\mathrm{d}x$$

c.
$$\int \frac{2x+1}{x^3+3x^2}\,\mathrm{d}x$$

Solution exercise 4a.
We start by finding the partial fraction expansion of the function we are integrating. We must find the constants A, B and C such that.

$$\frac{2x^2+5x+11}{(x+2)^2(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{x-1}$$

By calculating the right hand side above we see that

$$\begin{aligned}\frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{x-1} &= \frac{A(x+2)(x-1) + B(x-1) + C(x+2)^2}{(x+2)^2(x-1)} \\ &= \frac{x^2(A+C) + x(A+B+4C) + (-2A-B+4C)}{(x+1)^2(x-1)}.\end{aligned}$$

We must solve the following system of equations

$$\begin{aligned}A + C &= 2 \\ A + B + 4C &= 5 \\ -2A - B + 4C &= 11\end{aligned}$$

Which has the solution $A = 0$, $B = -3$ and $C = 2$. Thus it follows that

$$\frac{2x^2 + 5x + 11}{(x+2)^2(x-1)} = -\frac{3}{(x+2)^2} + \frac{2}{x-1}.$$

We can now calculate the integral:

$$\begin{aligned}\int \frac{2x^2 + 5x + 11}{(x+2)^2(x-1)}\,\mathrm{d}x &= \int -\frac{3}{(x+2)^2} + \frac{2}{x-1}\,\mathrm{d}x \\ &= \frac{3}{x+2} + 2\ln|x-1| + C.\end{aligned}$$

Solution exercise 4b

We start by finding the partial fraction expansion of the function we are integrating.

$$\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}\,\mathrm{d}x = \frac{A(x+1) + B(x-2)}{(x+1)(x-2)} = \frac{(A+B)x + A - 2B}{(x+1)(x-2)}.$$

We solve the system of equations $A + B = 0$ and $A - 2B = 1$ and see that $A = -\frac{1}{3}$ and $B = \frac{1}{3}$. Thus it follows that

$$\int \frac{1}{(x-2)(x+1)}\,\mathrm{d}x = \int -\frac{1}{3(x-2)} + \frac{1}{3(x+1)}\,\mathrm{d}x = -\frac{1}{3}\Big(\ln|x-2| - \ln|x+1|\Big) + C$$

Solution exercise 4c

We start by finding the partial fraction decomposition:

$$\begin{aligned}\frac{2x+1}{x^3+3x^2} &= \frac{2x+1}{x^2(x+3)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+3)} \\ &= \frac{Ax(x+3) + B(x+3) + Cx^2}{x^2(x+3)} \\ &= \frac{x^2(A+C) + x(3A+B) + 3B}{x^2(x+3)}\end{aligned}$$

We see that $A = \frac{5}{9}$, $B = \frac{1}{3}$ and $C = -\frac{5}{9}$. We calculate the integral:

$$\int \frac{2x+1}{x^3+3x^2}\,\mathrm{d}x = \int \frac{5}{9x} + \frac{1}{3x^2} - \frac{5}{9(x+3)}\,\mathrm{d}x = \frac{5}{9}\ln|x| - \frac{1}{3x} - \frac{5}{9}\ln|x+3| + C$$

$\boxed{5}$ Evaluate the following integrals.

a.

$$\int \frac{3x^2 - 5x + 3}{(x^2+1)^2}\,\mathrm{d}x$$

b.

$$\int \frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)}\,\mathrm{d}x$$

c.
$$\int \frac{1}{(x^2+4)(1+x^2)}\,\mathrm{d}x$$

Solution exercise 5a

We start by finding the partial fraction decomposition. We obsevere that the denumerator consists of the repeated linear factor $1+x^2$, and must thus find the constants A, B, C and D such that
$$\frac{3x^2-5x+3}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2}.$$
By calculating the right hand side of the equation above we see that
$$\frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} = \frac{(Ax+B)(x^2+1)+Cx+D}{(1+x^2)^2} = \frac{Ax^3+Bx^2+Ax+Cx+B+D}{(x^2+1)^2}$$
We solve the system of equations
$$\begin{aligned} A &= 0 \\ B &= 3 \\ A+C &= -5 \\ B+D &= 3, \end{aligned}$$
and see that$A=0$, $B=3$, $C=-5$ and $D=0$. Thus it follows that
$$\begin{aligned} \int \frac{3x^2-5x+3}{(x^2+1)^2}\,\mathrm{d}x &= \int \frac{3}{(x^2+1)} - \frac{5x}{(x^2+1)^2}\,\mathrm{d}x \\ &= 3\arctan(x) - \int \frac{5}{2u^2}du \\ &= 3\arctan(x) + \frac{5}{2u} + C \\ &= 3\arctan(x) + \frac{5}{2(x^2+1)} + C \end{aligned}$$
Here we have used the substitution $u = x^2+1$.

Solution exercise 5b

We start by finding the partial fraction decomposition. We obsevere that the denumerator consists of two distinct irreducible factors, and we start by finding the constants A, B, C and D such that
$$\frac{x^3-x^2+x-4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} = \frac{Ax^3+Cx^3+Bx^2+Dx^2+4Ax+Cx+4B+D}{(x^2+1)(x^2+4)}.$$
We solve the system of equations
$$\begin{aligned} A+C &= 1 \\ B+D &= -1 \\ 4A+C &= 1 \\ 4B+D &= -4, \end{aligned}$$

and see that$A = 0$, $B = -1$, $C = 1$ and $D = 0$. Now, we can calculate the integral:

$$\int \frac{x^3 - x^2 + x - 4}{(x^2+1)(x^2+4)}\,\mathrm{d}x = \int -\frac{1}{(x^2+1)} + \frac{x}{x^2+4}\,\mathrm{d}x = -\arctan(x) + \frac{1}{2}\ln|x^2+4| + C.$$

For the last integral we have used the substitution $u = x^2 + 4$.

Solution exercise 5c.
We start by finding constants A, B, C and D such that

$$\frac{1}{(x^2+4)(x^2+1)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1} = \frac{x^3(A+C) + x^2(B+D) + x(A+4C) + B + 4D}{(x^2+4)(x^2+1)}.$$

We solve the system of equations

$$\begin{aligned} A + C &= 0 \\ B + D &= 0 \\ A + 4C &= 0 \\ B + 4D &= 1, \end{aligned}$$

and see that $A = 0$, $B = -\frac{1}{3}$, $C = 0$ and $D = \frac{1}{3}$. Now, we can calculate the integral:

$$\begin{aligned} \int \frac{1}{(x^2+4)(x^2+1)}\,\mathrm{d}x &= \int -\frac{1}{3(x^2+4)} + \frac{1}{3(x^2+1)}\,\mathrm{d}x = -\frac{1}{6}\int \frac{1}{u^2+1} + \frac{1}{3}\arctan(x) + C \\ &= -\frac{1}{6}\arctan(u) + \frac{1}{3}\arctan(x) + C = -\frac{1}{6}\arctan(\frac{x}{2}) + \frac{1}{3}\arctan(x) + C, \end{aligned}$$

where in the third equality we used the substitution $u = \frac{x}{2}$.