MA0002 Brukerkurs i Matematikk B
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Exercise set 4

1 Let $N(t)$ be the size of a population after t years. Assume that

$$\frac{dN}{dt} = 0.8N\left(1 - \frac{N}{27}\right),$$

and that $N(0) = 2$. Solve the differential equation, and find $\lim_{t\to\infty} N(t)$. What does the limit value tell you?

Solution. We know that a general equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

with initial condition $N(0) = N_0$ have the solution

$$N(t) = \frac{K}{1 + (K/N_0 - 1)e^{-rt}}.$$

This is equation (8.33) at page 401 in the book. Thus our solution is

$$N(t) = \frac{27}{1 + (27/2 - 1)e^{-0.8t}} = \frac{27}{1 + 13.5e^{-0.8t}}.$$

Further, we see that $\lim_{N\to\infty} N(t) = K = 27$. This tells us that in the long run the population will consist of 27 members.

2 Let

$$\frac{dy}{dx} = (2 - y)(3 - y).$$

Find the point equilibria and determine the stability of the equilibria.

Solution. We let $g(y) = (2 - y)(3 - y)$. To find the equilibria we must solve the equation $g(y) = 0$. We see that we have two equlibrium; $y_1 = 2$ and $y_2 = 3$. To decide the stability we calculate $g'(y) = 2y - 5$. Thus we can see that $g'(2) = -1 < 0$ and that $g'(3) = 1 > 0$. It follows that $y_1 = 2$ is locally stable and that $y_2 = 3$ is unstable.

3 Assume that $N(t)$ is the size of a population at time t and that N satisfies the differential equation Let

$$g(N) = N\left(1 - \frac{N}{50}\right) - \frac{9N}{5 + N}.$$

(a) Sketch the graph of g.

(b) Find the equilibria of the differential equation.

(c) Determine the stability of the equilibria.

Solution.

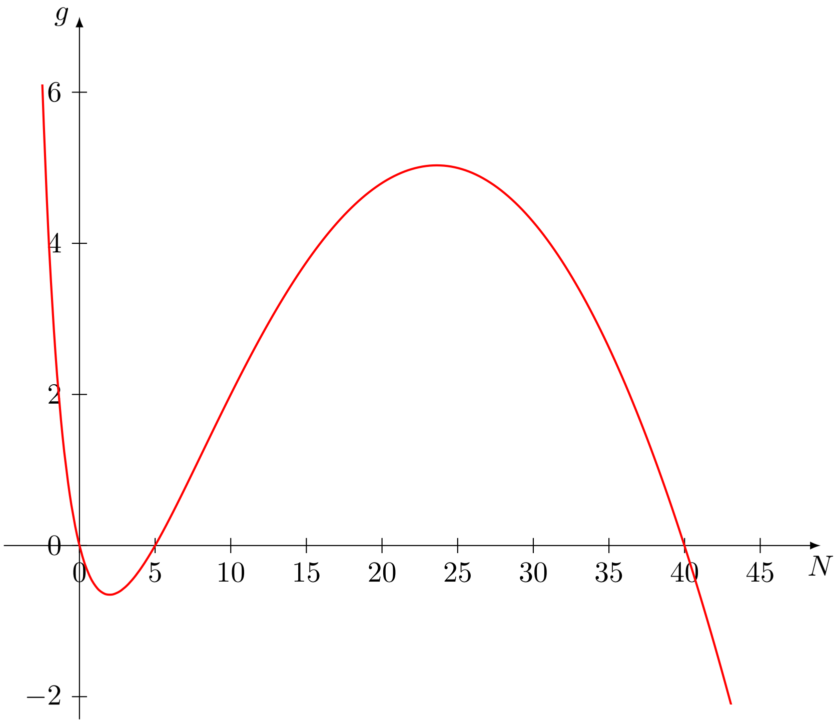

Figure 1: The graph of g

(a)

(b) By inspecting the figure we see that we have two equilibrium; $y_1(x) = 0$, $y_2(x) = 5$ and $y_3(x) = 40$. We can also confirme this by mathematical analysis:

$$\begin{aligned} g(N) &= N\left(1 - \frac{N}{50}\right) - \frac{9N}{5+N} \\ &= N\left(1 - \frac{N}{50} - \frac{9}{5+N}\right) \\ &= N\frac{50(5+N) - N(5+N) - 450}{50(5+N)} \\ &= -\frac{N(N^2 - 45N + 200)}{50(5+N)} \\ &= -\frac{N(N-5)(N-40)}{50(5+N)} \end{aligned}$$

and thus

$$g(N) = 0 \qquad \iff \qquad N = 0,\ N = 5 \text{ or } N = 40.$$

(c) By inspecting the figure we see that y_1 and y_3 are stable,while y_2 is unstable. We can also confirm this by mathematical analysis:

$$\begin{aligned} g'(N) &= 1 - \frac{N}{25} - \frac{9(5+N) - 9N}{(5+N)^2} \\ &= 1 - \frac{N}{25} - \frac{45}{(5+N)^2}, \end{aligned}$$

so

$$\begin{aligned} g'(0) &= 1 - 45/5^2 \\ &= -4/5 < 0. \\ g'(5) &= 1 - 5/25 - 45/10^2 \\ &= 7/20 > 0. \\ g'(40) &= 1 - 40/25 - 45/45^2 \\ &= -28/45 < 0. \end{aligned}$$

$\boxed{4}$ Assume the single-compartment model defined in subsection 8.2.2. That is assume that $C(t)$ is the concentration of the solute at time t and assume that

$$\frac{dC}{dt} = 5(4 - C(t)).$$

(a) Solve the equation above when $C(0) = 2$.

(b) Find $\lim_{t\to\infty} C(t)$.

(c) Use your answer from (a) to find t when $C(t) = 10$.

Solution.

(a) We solve the seprable differential equation:

$$\begin{aligned} \frac{dC}{dt} &= 5(4 - C(t)) \\ \frac{dC}{4 - C(t)} &= 5dt \\ \int \frac{dC}{4 - C(t)} &= \int 5dt \\ -\ln|4 - C(t)| &= 5t + C_1 \\ \ln|4 - C(t)| &= -5t - C_1 \\ 4 - C(t) &= C_2 e^{-5t} \\ C(t) &= 4 - C_2 e^{-5t} \end{aligned}$$

Then we use the initial conditon to find the constant C_2:

$$\begin{aligned} C(0) = 2 &= 4 - C_2 \\ \implies C_2 &= 2 \end{aligned}$$

We thus obtain the solution:

$$C(t) = 4 - 2e^{-5t}$$

(b) We see that

$$\lim_{t\to\infty} C(t) = \lim_{t\to\infty} 4 - 2e^{-5t}$$
$$= 4$$

(c) We solve the equation $C(t) = 3$.

$$\begin{aligned}
3 &= 4 - 2e^{-5t} \\
-1 &= -2e^{-5t} \\
e^{-5t} &= \frac{1}{2} \\
-5t &= \ln\frac{1}{2} \\
t &= -\frac{1}{5}\ln\frac{1}{2} = \frac{\ln 2}{5}
\end{aligned}$$

That is: $C = 3$ for $t = \frac{\ln 2}{5}$.

$\boxed{5}$ Let $N(t)$ be the size of a population at the time t. Assume that

$$\frac{dN}{dt} = 5N(N-2)\left(1 - \frac{N}{10}\right),$$

for $t \geq 0$.

(a) Find all the equilibria for this differential equation.

(b) Determine the satbility of the equilibrium.

Solution.

(a) Leting

$$G(N) = 5N(N-2)\left(1 - \frac{N}{10}\right)$$

and solving the equation $G(N) = 0$ we see that we obtain the three solutions; $N_1 = 0$, $N_2 = 2$ and $N_3 = 10$.

(b) To determine the stability we start by finding

$$G'(N) = -\frac{3N^2}{2} + 12N - 10.$$

We see that

$$\begin{aligned}
G'(0) &= -10 < 0 \\
G'(2) &= 8 > 0 \\
G'(10) &= -40 < 0.
\end{aligned}$$

Thus $N_1 = 0$ and $N_3 = 10$ are localy stable while, $N_2 = 2$ is unstable.

$\boxed{6}$ Assume the classical Kermack–McKendrick model for spread of disease (see chapter 8.3.1 in the book). Decide whether the disease in the 2 cases below will spread or not by determening if R_0 is greater than, or less than 1.

(a) $S(0) = 1000$, $b = 0.4$, $a = 300$

(b) $S(0) = 500$, $b = 0.1$, $a = 200$.

Solution

(a) We see that $R_0 = \frac{bS(0)}{a} = \frac{0.4*1000}{300} > 1$, so the disease will spread

(b) We see that $R_0 = \frac{0.1*500}{200} < 1$, so the disease will not spread.