MA 0003
14/11

$\boxed{\text{Matriser til en lineær transformasjon}}$

Lay 1.9

La $T: \mathbb{R}^n \to \mathbb{R}^m$ være en lineær transformasjon

dvs $\begin{aligned} T(\underline{u}+\underline{v}) &= T(\underline{u})+T(\underline{v}) \\ T(c\underline{u}) &= cT(\underline{u}) \end{aligned} \quad \Rightarrow \quad T(c\underline{u}+d\underline{v}) = cT(\underline{u})+dT(\underline{v})$

for alle $\underline{u},\underline{v} \in \mathbb{R}^n \quad c,d \in \mathbb{R}$

eks. $n=m=2$

$$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = T\left(3\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 3T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = T\left(a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix}\right) = aT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + bT\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

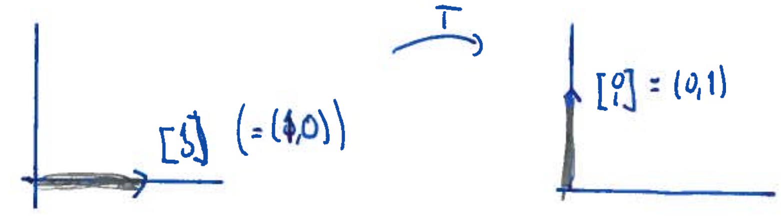
Altså: hvis $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ og $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ er kjent, kan

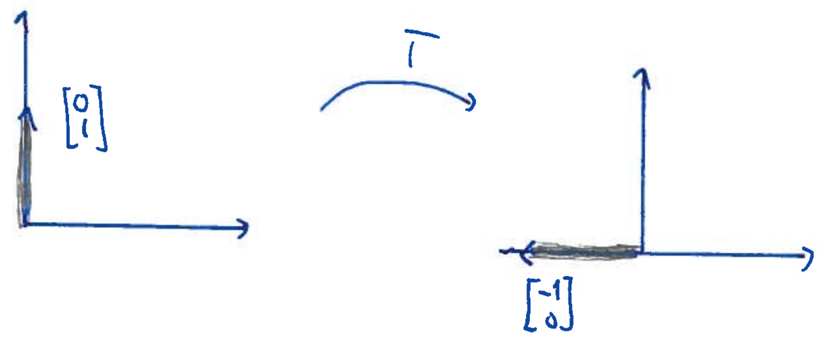
vi beregne $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)$ for alle $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$

spesielt; vi kan beregne $T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$

Eks.

Rotasjon $90°$ $\left(= \frac{\pi}{2}\right)$





$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

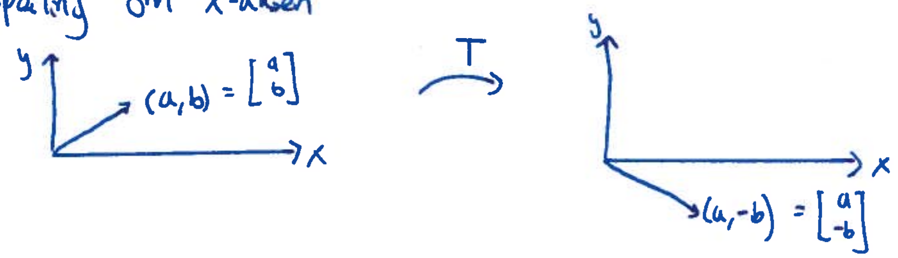
$$\Rightarrow \quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + b\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

NB! $\quad a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

så $\quad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$

Eks: Speiling om x-aksen



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + b\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ -b \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

Teorem

$T : \mathbb{R}^n \to \mathbb{R}^m$, en lineær transformasjon

La $A = \left[T\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \;\vdots\; T\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix} \;\vdots\; \cdots \;\vdots\; T\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix} \right]$

dvs A er matrisa med kolonner $T\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, T\begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, T\begin{bmatrix}0\\\vdots\\1\end{bmatrix}$

Da er $T(\underline{x}) = A\underline{x}$ for alle $\underline{x}$ i $\mathbb{R}^n$

A kalles standardmatrisa til T.

Eks.

La $T : \mathbb{R}^3 \to \mathbb{R}^3$ være en lin. trans. slik at

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$$

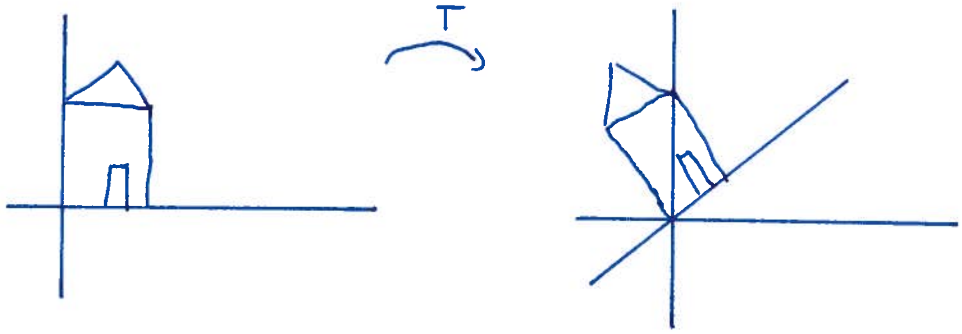
Da er $T(\underline{x}) = A\underline{x}$

der $A = \begin{bmatrix}2 & 1 & -1\\1 & 4 & 0\\3 & 3 & -1\end{bmatrix}$

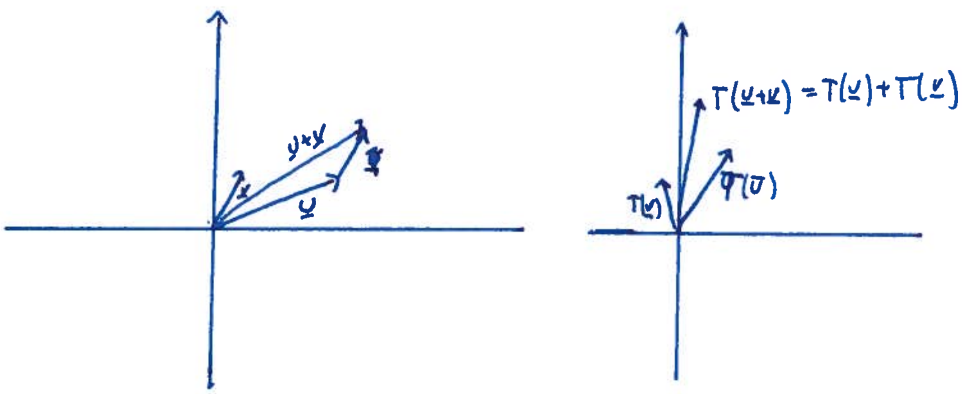
altså $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2 & 1 & -1\\1 & 4 & 0\\3 & 3 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2x+y-z\\x+4y\\3x+3y-z\end{bmatrix}$

Eks

rotasjon 45° $\left(= \frac{\pi}{4}\right)$



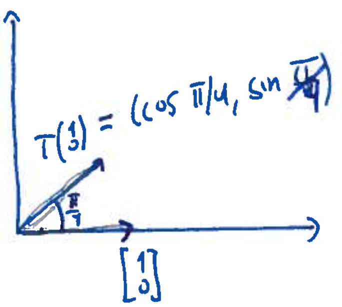
Rotasjon er en lineær transformasjon



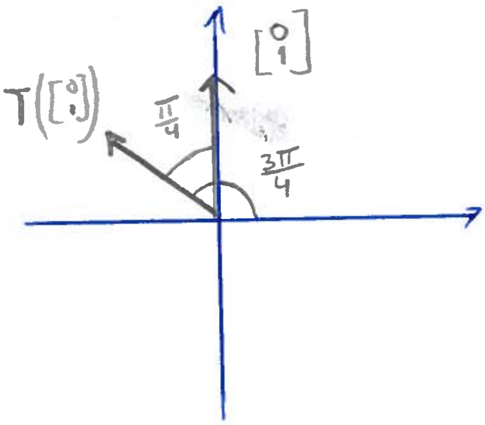
(bedre figur, side 78, figur 6, i Lay)

Vi trenger å finne

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ og $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$



$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\frac{\pi}{4}\\ \sin\frac{\pi}{4}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

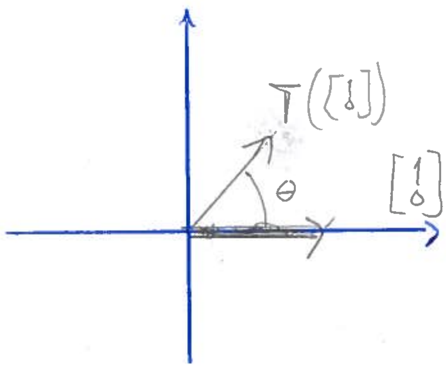


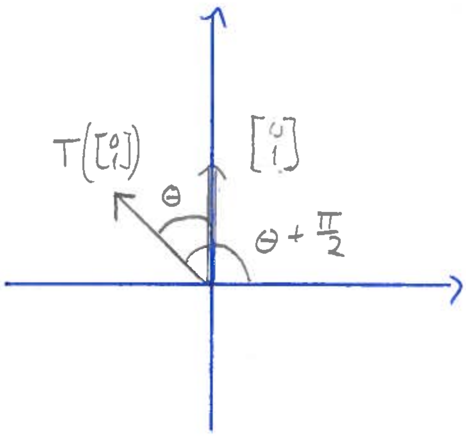
$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\cos \frac{3}{4}\pi \\ \sin \frac{3}{4}\pi\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

Rotasjon

Har sett: rotasjon $\frac{\pi}{2}, \frac{\pi}{4}$ (altså $90°, 45°$)

Generelt, rotasjon θ grader $\theta \in [0, 2\pi)$





$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos \theta \\ \sin \theta\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\cos (\theta + \frac{\pi}{2}) \\ \sin (\theta + \frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}-\sin \theta \\ \cos \theta\end{bmatrix}$$

oppgave

Dermed blir standard matrise

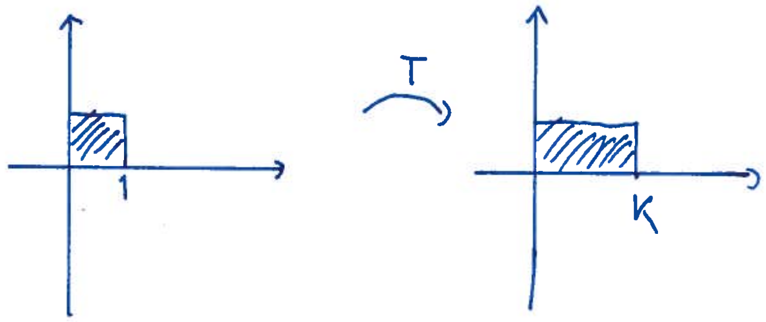
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

spes $\quad \theta = 90^\circ \; \left(= \frac{\pi}{2}\right) \quad \Rightarrow \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

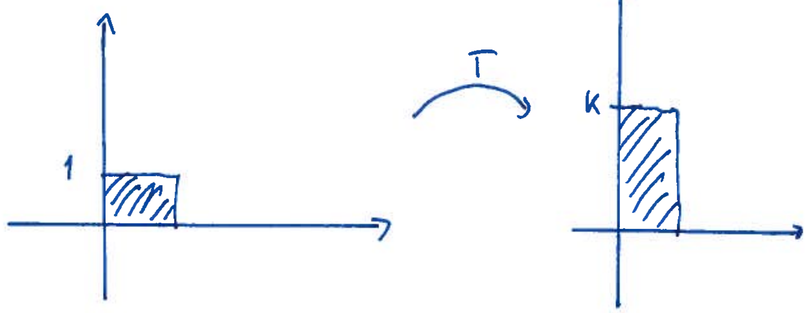
$\theta = 45^\circ \; \left(= \frac{\pi}{4}\right) \quad \Rightarrow \quad A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Ekspansjon/Kontraksjon

Ekspansjon i x-retning $\qquad A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \quad k > 1$



Ekspansjon: y-retning $\qquad A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \quad k > 1$

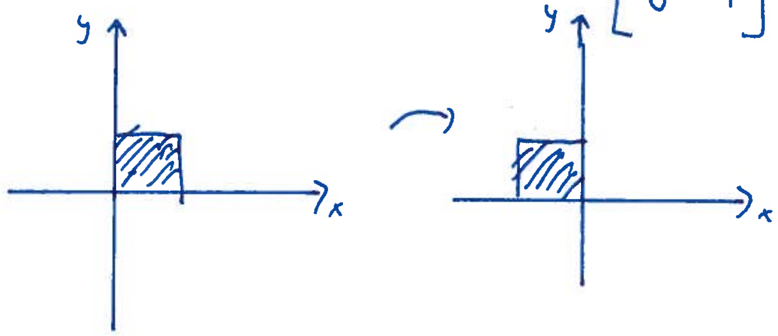


Kontraksjon i x-retning : $A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ $\quad 0 \leq k < 1$

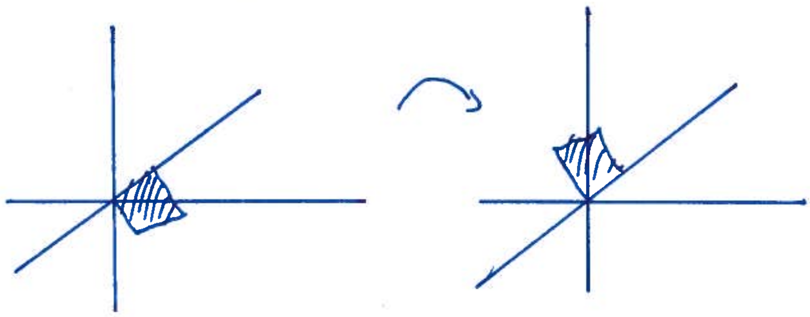
Refleksjon

Refleksjon om x-aksen $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Refleksjon om y-aksen $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$



Refleksjon om linja $y = x$ $\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

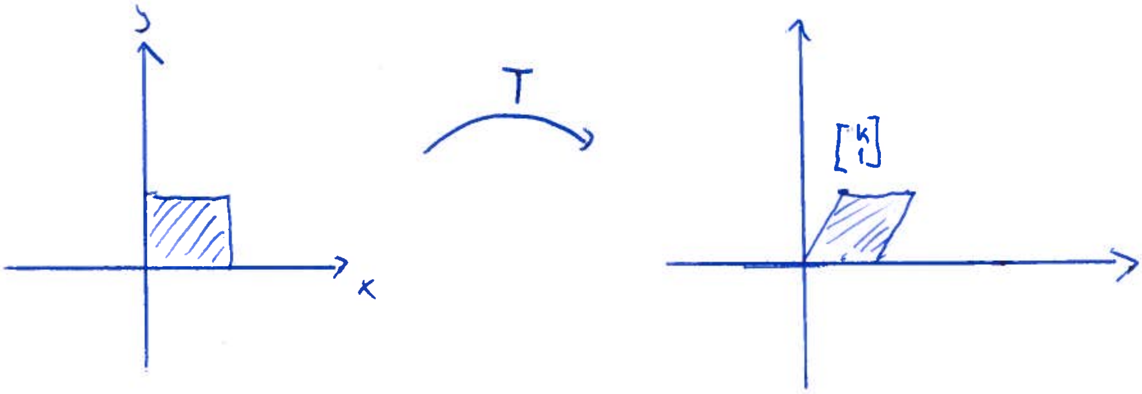


Forskyvning (shear)

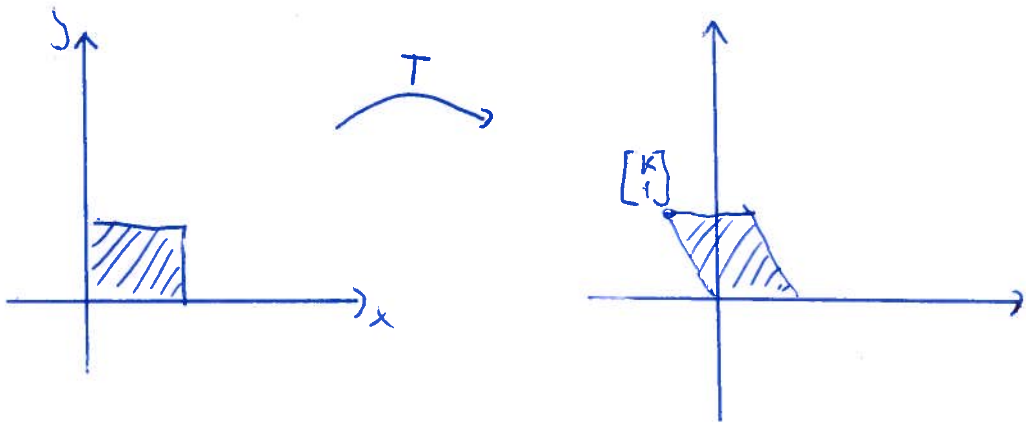
Husk

$T(\underline{u}) = A\underline{u}$

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad k > 0$$



$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad k < 0$$



Projeksjoner

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

projeksjon på x-aksen

$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ y \end{bmatrix}$

projeksjon på y-aksen