NTNU
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Department of Mathematical Sciences

Examination paper for

MA0301 Elementary discrete mathematics

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Examination date: 24 May 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Language: English

Number of pages: 5

Number of pages enclosed: 0

Informasjon om trykking av eksamensoppgave	
Originalen er:	
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Date Signature

Exercise 1 (Sets):	10 points
Exercise 2 (Relations):	20 points
Exercise 3 (Induction):	20 points
Exercise 4 (Functions):	15 points
Exercise 5 (Boolean algebra):	10 points
Exercise 6 (Languages and finite state automata):	10 points
Exercise 7 (Graphs):	15 points

Note: In each of 1.1, 2.1, 4.1, and 7.1, at least one and possibly several of the alternatives are correct.

Problem 1 **Sets** (10 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. $|\mathcal{P}(A)| = 128$.

   II) The set $A = \{1, 2, 3, 4, 5, 6, 7\}$ has 127 nonempty subsets.

   III) The set $A = \{1, 2, 3, 4, 5, 6, 7\}$ has 127 proper subsets.

2. **(7 points)** Let A, B be two sets. We assume that $(A \times A) = (B \times B)$. Show that $A = B$.

   Recall that the cartesian product of two arbitrary sets X and Y is $X \times Y := \{(x, y) | \; x \in X, \; y \in Y\}$.

Problem 2 **Relations** (20 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) A relation R on a set A is an equivalence relation if and only if R is reflexive, anti-symmetric, and transitive.

II) A relation R on a set A is an equivalence relation if and only if R is reflexive, symmetric, and transitive.

III) A relation R on a set A is an equivalence relation if and only if R is anti-reflexive, symmetric, and transitive.

2. **(5 points)** Let $A := \{1,2,3,4\}$. Define the relation

$$R = \{(1,1),(2,2),(3,4),(3,3),(4,4)\}.$$

Prove or disprove that R is an equivalence relation.

3. **(6 points)** We define the following relation R on $\mathbb{N}\times\mathbb{N}$: $(a,b)R(x,y)$ if and only if $ay = bx$. Show that R is an equivalence relation.

4. **(6 points)** Consider the "divides" relation on the set $A = \{1,2,4,5,10,15,20\}$, i.e., xRy if and only if x divides y (that is, $y = zx$ for some $z \in \mathbb{Z}$). Draw the Hasse diagram.

Problem 3 **Induction** (20 points)

1. **(8 points)** Use induction to show that

$$\sum_{k=1}^{n} k(k-1)(k-2) = \frac{1}{4}(n+1)n(n-1)(n-2).$$

2. **(12 points)** Recall that the Lucas numbers are defined recursively, $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n > 1$. Show by induction that for $n > 0$,

$$\sum_{i=1}^{n} iL_i = nL_{n+2} - L_{n+3} + 4.$$

Problem 4 **Functions** (15 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) A function $f\colon A \to B$ is injective if and only if each element of B appears at most once as the image of an element of A.

   II) A function $f\colon A \to B$ is injective if and only if for all $a_1, a_2 \in A$, $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$.

   III) A function $f\colon A \to B$ is injective if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

2. **(5 points)** Define the function $f(x) := \frac{x+1}{x-1}$ with domain $\mathbb{R}-\{1\}$ and codomain $\mathbb{R}-\{1\}$. Calculate $(f \circ f)(x)$ and draw a conclusion.

3. **(7 points)** The function $f(x) = \frac{x}{x^2+1}$ is defined on $\mathbb{R}$. Prove or disprove that f is injective.

Problem 5 **Boolean algebra** (10 points)

1. **(5 points)** Use Boolean algebra to simplify the expression

$$V = \bar{x}\bar{y}z + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z} + xyz.$$

   (Hint: The identity $a\bar{b} + a = b + a$ may be helpful.)

2. **(5 points)** Use Boolean algebra to simplify the expression

$$W = (x + y + \bar{z})(\bar{x} + y + z)(\bar{x} + y + \bar{z})(\bar{x} + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z}).$$

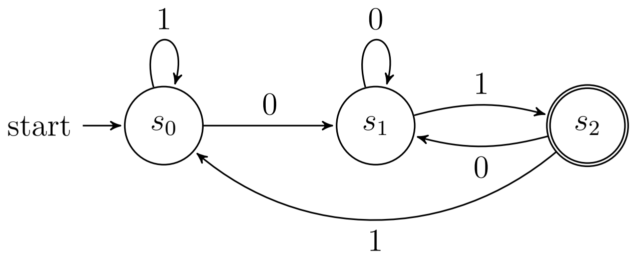

Figure 1: The automaton A.

Problem 6 Languages and finite state automata (10 points)

1. **(5 points)** Let $\Sigma := \{0, 1\}$ be an alphabet. Find a regular expression that defines the language $L \subset \Sigma^*$ consisting of all strings of 0's and 1's with an odd number of 1's.

2. **(5 points)** i) What is the state table $T(A)$ for the automaton A in Figure 1?

   ii) Determine the arrival state for each of the input sequences

   a) 01 b) 0011 c) 010101

   iii) What is the language $L(A)$ accepted by A?

Problem 7 Graphs (15 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v - 6$.

   II) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v + 6$.

   III) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \geq 2e$ and $e \geq 3v + 6$.

2. **(6 points)** If $G = (V, E)$ is connected graph with $|E| = 17$ and $\deg(v) > 2$ for all $v \in V$, what is the maximum value of $|V|$?

3. **(6 points)** Let $G = (V, E)$ be an undirected connected loop-free graph. Suppose that G is planar and determines $r = 53$ regions. If for some planar embedding of G, each region has at least five edges in its boundary, show that $|V| > 81$.