NTNU
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Examination paper for
MA0301 Elementary discrete mathematics

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Examination date: 24 May 2018

Examination time (from–to): 09:00–13:00

Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Language: English

Number of pages: 7

Number of pages enclosed: 0

Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig ☐ **2-sidig** ☒

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Exercise 1 (Sets):	10 points
Exercise 2 (Relations):	20 points
Exercise 3 (Induction):	20 points
Exercise 4 (Functions):	15 points
Exercise 5 (Boolean algebra):	10 points
Exercise 6 (Languages and finite state automata):	10 points
Exercise 7 (Graphs):	15 points

Note: In each of 1.1, 2.1, 4.1, and 7.1, at least one and possibly several of the alternatives are correct.

Problem 1 **Sets** (10 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. $|\mathcal{P}(A)| = 128$.

   II) The set $A = \{1, 2, 3, 4, 5, 6, 7\}$ has 127 nonempty subsets.

   III) The set $A = \{1, 2, 3, 4, 5, 6, 7\}$ has 127 proper subsets.

2. **(7 points)** Let A, B be two sets. We assume that $(A \times A) = (B \times B)$. Show that $A = B$.

   Recall that the cartesian product of two arbitrary sets X and Y is $X \times Y := \{(x, y) | \, x \in X, \; y \in Y\}$.

Solution 1 1) All three statements, i.e., I), II), and III), are true.

2) We work with the assumption that $(A \times A) = (B \times B)$ and want to show that $A = B$, i.e., $A \subseteq B$ and $B \subseteq A$. Let $x \in A$ then $(x, x) \in A \times A$ By assumption this implies that $(x, x) \in B \times B$. Therefore $x \in B$ and $A \subseteq B$. The other direction works completely analogously.

Problem 2 **Relations** (20 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) A relation R on a set A is an equivalence relation if and only if R is reflexive, anti-symmetric, and transitive.

   II) A relation R on a set A is an equivalence relation if and only if R is reflexive, symmetric, and transitive.

   III) A relation R on a set A is an equivalence relation if and only if R is anti-reflexive, symmetric, and transitive.

2. **(5 points)** Let $A := \{1, 2, 3, 4\}$. Define the relation

$$R = \{(1, 1), (2, 2), (3, 4), (3, 3), (4, 4)\}.$$

   Prove or disprove that R is an equivalence relation.

3. **(6 points)** We define the following relation R on $\mathbb{N} \times \mathbb{N}$: $(a, b)R(x, y)$ if and only if $ay = bx$. Show that R is an equivalence relation.

4. **(6 points)** Consider the "divides" relation on the set $A = \{1, 2, 4, 5, 10, 15, 20\}$, i.e., xRy if and only if x divides y (that is, $y = zx$ for some $z \in \mathbb{Z}$). Draw the Hasse diagram.

Solution 2 1) II) is correct.

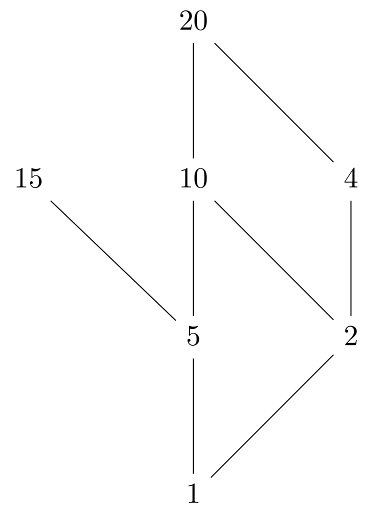
2) R is reflexive and transitive, but not symmetric, as $(3, 4)$ is included, while $(4, 3)$ is not.

3) R is reflexive: for $(x, y) \in \mathbb{N} \times \mathbb{N}$, we have that $(x, y)R(x, y)$, as $xy = yx$.

R is symmetric: assume that $(x, y)R(u, v)$. Then $xv = yu$, which is the same as $uy = vx$. This implies that $(u, v)R(x, y)$.

R is transitive: assume that $(u, v)R(x, y)$ and $(x, y)R(a, b)$. That means that $uy = vx$ and $xb = ya$. Therefore, $uyxb = vxya$, and this implies that $ub = va$, i.e., $(u, v)R(a, b)$.

4)



Problem 3 **Induction** (20 points)

1. **(8 points)** Use induction to show that
$$\sum_{k=1}^{n} k(k-1)(k-2) = \frac{1}{4}(n+1)n(n-1)(n-2).$$

2. **(12 points)** Recall that the Lucas numbers are defined recursively, $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$, for $n > 1$. Show by induction that for $n > 0$,
$$\sum_{i=1}^{n} iL_i = nL_{n+2} - L_{n+3} + 4.$$

Solution 3 1) Basis step: $n = 1$: on the left-hand side we have zero; on the right-hand side we also have zero. Induction hypothesis: we have for $i > 0$ that $\sum_{k=1}^{i} k(k-1)(k-2) = \frac{1}{4}(i+1)i(i-1)(i-2)$. Induction step:

$$\begin{aligned}\sum_{k=1}^{i+1} k(k-1)(k-2) &= \sum_{k=1}^{i} k(k-1)(k-2) + (i+1)i(i-1) \\ &= \frac{1}{4}(i+1)i(i-1)(i-2) + (i+1)i(i-1) \\ &= \frac{1}{4}(i+1)i(i-1)(i-2+4) \\ &= \frac{1}{4}(i+1)i(i-1)(i+2),\end{aligned}$$

which is what we wanted to show.

2) Basis step: $n = 1$: we have on the left-hand side $1L_1 = 1$; on the right-hand side we have $1L_3 - L_4 + 4 = 4 - 7 + 4 = 1$. Induction hypothesis: we have for $k > 0$ that $\sum_{i=1}^{k} iL_i = kL_{k+2} - L_{k+3} + 4$.. For $k+1$ we get

$$\begin{aligned}\sum_{i=1}^{k+1} iL_i &= \sum_{i=1}^{k} iL_i + (k+1)L_{k+1} \\ &= kL_{k+2} - L_{k+3} + 4 + (k+1)L_{k+1}. \\ &= (k+1)L_{k+2} - L_{k+2} - L_{k+3} + 4 + (k+1)L_{k+1} \\ &= (k+1)L_{k+2} + (k+1)L_{k+1} - L_{k+2} - L_{k+3} + 4 \\ &= (k+1)(L_{k+2} + L_{k+1}) - (L_{k+2} + L_{k+3}) + 4 \\ &= (k+1)L_{k+3} - L_{k+4} + 4 \\ &= (k+1)L_{(k+1)+2} - L_{(k+1)+3} + 4\end{aligned}$$

Problem 4 **Functions** (15 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) A function $f\colon A \to B$ is injective if and only if each element of B appears at most once as the image of an element of A.

   II) A function $f\colon A \to B$ is injective if and only if for all $a_1, a_2 \in A$, $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$.

   III) A function $f\colon A \to B$ is injective if and only if for all $a_1, a_2 \in A$, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

2. **(5 points)** Define the function $f(x) := \frac{x+1}{x-1}$ with domain $\mathbb{R} - \{1\}$ and codomain $\mathbb{R} - \{1\}$. Calculate $(f \circ f)(x)$ and draw a conclusion.

3. **(7 points)** The function $f(x) = \frac{x}{x^2+1}$ is defined on $\mathbb{R}$. Prove or disprove that f is injective.

Solution 4 1) I) and III) are correct. $a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$ holds for all functions f.

2)

$$(f \circ f)(x) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x$$

The composition of the function with itself gives the identity, i.e., it is its own inverse.

3) The function is not injective. Indeed, note that

$$f(x_1) = \frac{x_1}{x_1^2 + 1} = f(x_2) = \frac{x_2}{x_2^2 + 1}$$

implies that $x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$ and this implies that $x_1 x_2 (x_1 - x_2) = x_1 - x_2$. The latter tells us that either $x_1 = x_2$ or $x_1 x_2 = 1$. We try choosing x_1 different from x_2, but $x_1 x_2 = 1$, e.g. $x_1 = 4$ and $x_2 = 1/4$. This gives a counterexample, as $f(4) = 4/17 = f(1/4)$.

Problem 5 Boolean algebra (10 points)

1. **(5 points)** Use Boolean algebra to simplify the expression
$$V = \bar{x}\bar{y}z + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z} + xyz.$$
(Hint: The identity $a\bar{b} + a = b + a$ may be helpful.)

2. **(5 points)** Use Boolean algebra to simplify the expression
$$W = (x + y + \bar{z})(\bar{x} + y + z)(\bar{x} + y + \bar{z})(\bar{x} + \bar{y} + z)(\bar{x} + \bar{y} + \bar{z}).$$

Solution 5 1)

$$\begin{aligned} V &= \bar{x}\bar{y}z + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z} + xyz \\ &= \bar{x}\bar{y}z + x\bar{y}(\bar{z} + z) + xy(\bar{z} + z) \\ &= \bar{x}\bar{y}z + x\bar{y} + xy \\ &= \bar{x}\bar{y}z + x(\bar{y} + y) \\ &= \bar{x}\bar{y}z + x \\ &= \bar{y}z + x. \end{aligned}$$

3) $W = \bar{V} = (y + \bar{z})x$.

Problem 6 Languages and finite state automata (10 points)

1. **(5 points)** Let $\Sigma := \{0, 1\}$ be an alphabet. Find a regular expression that defines the language $L \subset \Sigma^*$ consisting of all strings of 0's and 1's with an odd number of 1's.

2. **(5 points)** i) What is the state table $T(A)$ for the automaton A in Figure 1?

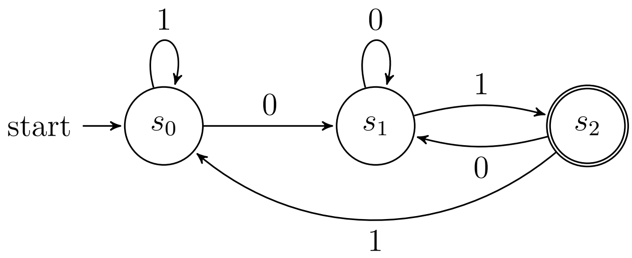


Figure 1: The automaton A.

ii) Determine the arrival state for each of the input sequences

$a)$ 01 $b)$ 0011 $c)$ 010101

iii) What is the language $L(A)$ accepted by A?

Solution 6 1) 0*10*(0*10*10*)*

2) i)

A	ν	
	0	1
s_0	s_1	s_0
s_1	s_1	s_2
s_2	s_1	s_0

ii) s_2, s_0, s_2

iii) All strings that end in 01.

Problem 7 Graphs (15 points)

1. **(3 points)** Which of the three statements is/are correct?

   I) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v - 6$.

   II) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \leq 2e$ and $e \leq 3v + 6$.

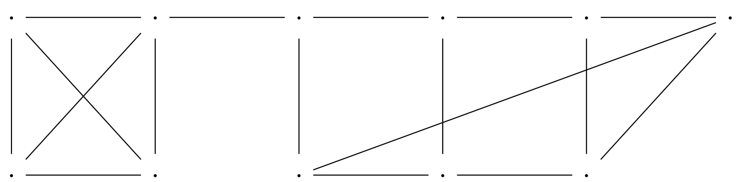
   III) Let $G = (V, E)$ be a loop-free connected planar graph with $|V| = v$, $|E| = e > 2$, and r regions. Then $3r \geq 2e$ and $e \geq 3v + 6$.

2. **(6 points)** If $G = (V, E)$ is connected graph with $|E| = 17$ and $\deg(v) > 2$ for all $v \in V$, what is the maximum value of $|V|$?

3. **(6 points)** Let $G = (V, E)$ be an undirected connected loop-free graph. Suppose that G is planar and determines $r = 53$ regions. If for some planar embedding of G, each region has at least five edges in its boundary, show that $|V| > 81$.

Solution 7 1) I) and II) are correct.

2) $2|E| = 34 = \sum_{v \in V} \deg(v) \geq 3|V|$. Therefore, the maximum value of $|V|$ is at most 11. The graph below shows that this value is attainable, so it is indeed the maximum.



3) We have $r = 53$ regions and we know that each region has at least five edges in its boundary. Therefore $2|E| > 5 \cdot 53$. We also know that $|V| = |E| - 53 + 2$. Therefore $|E| - 51 \geq \frac{1}{2} \cdot 5 \cdot 53 - 51 = \frac{265-102}{2} = \frac{163}{2} = 81.5$. This implies that $|V| > 81$.