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Solutions to exercise set 5

$\boxed{1}$ **4.1.1a)** Let $S(k)$ be the statement that the equation holds for $n = k$. We need to show that $S(1)$ holds (the basis step), and that for all $k \geq 1$, if $S(k)$ holds, then $S(k+1)$ also holds (the inductive step). Putting $n = 1$ in the equation, we get $1^2 = 1 \cdot 1 \cdot 3/3$, which is true, so $S(1)$ is true. Suppose now that $S(k)$ holds. Then we get the following:

$$
\begin{aligned}
1^2 + 3^2 + \cdots + (2k-1)^2 &= \frac{k(2k-1)(2k+1)}{3} \\
\implies 1^2 + 3^2 + \cdots + (2k-1)^2 + (2k+1)^2 &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\
\implies 1^2 + 3^2 + \cdots + (2k+1)^2 &= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\
\implies 1^2 + 3^2 + \cdots + (2(k+1)-1)^2 &= \frac{(k(2k-1) + 3(2k+1))(2k+1)}{3} \\
&= \frac{(2k^2+5k+3)(2k+1)}{3} \\
&= \frac{(k+1)(2k+3)(2k+1)}{3} \\
&= \frac{(k+1)(2(k+1)+1)(2(k+1)-1)}{3}
\end{aligned}
$$

This is the equation with $n = k+1$, so we have proved that $S(k)$ implies $S(k+1)$, and we are done.

b) Basis step: Putting $n = 1$, we get $1 \cdot 3 = 1 \cdot 2 \cdot 9/6$, which is true.
Inductive step: Assume the equation holds for $n = k$, where $k \geq 1$ is a fixed number.

We get

$$\begin{aligned}
1\cdot 3+2\cdot 4+\cdots+k(k+2) &= \frac{k(k+1)(2k+7)}{6}\\
\implies 1\cdot 3+2\cdot 4+\cdots+k(k+2)+(k+1)(k+3) &= \frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)\\
\implies 1\cdot 3+2\cdot 4+\cdots+(k+1)((k+1)+2) &= \frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6}\\
&= \frac{(k+1)(k(2k+7)+6(k+3))}{6}\\
&= \frac{(k+1)(2k^2+13k+18)}{6}\\
&= \frac{(k+1)(k+2)(2k+9)}{6}\\
&= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}
\end{aligned}$$

This is the equation for $n=k+1$, so we have showed that if the equation is true for $n=k$, it is true for $n=k+1$.

c) Basis step: $n=1$ gives $1/(1\cdot 2)=1/2$, which is true.
Inductive step: We show that if the equation is true for $n=k$, then it is true for $n=k+1$:

$$\begin{aligned}
\sum_{i=1}^{k}\frac{1}{i(i+1)} &= \frac{k}{k+1}\\
\implies \frac{1}{(k+1)(k+2)}+\sum_{i=1}^{k}\frac{1}{i(i+1)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\
\implies \sum_{i=1}^{k+1}\frac{1}{i(i+1)} &= \frac{k(k+2)+1}{(k+1)(k+2)}\\
&= \frac{k^2+2k+1}{(k+1)(k+2)}\\
&= \frac{(k+1)^2}{(k+1)(k+2)}\\
&= \frac{k+1}{k+2}
\end{aligned}$$

$\boxed{2}$ We count the numbers that are divisible by 3, 5, or 7. First observe that the number of members of Y that are divisible by n is $\lfloor 600/n\rfloor$. (Here $\lfloor x\rfloor$ is the *floor function*, which returns the greatest integer less than or equal to x.) Thus there are $\lfloor 600/3\rfloor=200$ numbers in Y that are divisible by 3, $\lfloor 600/5\rfloor=120$ that are divisible by 5, and $\lfloor 600/7\rfloor=\lfloor 85.7\ldots\rfloor=85$ that are divisible by 7. But now we have counted some numbers twice, so we need to subtract $\lfloor 600/(3\cdot 5)\rfloor=40$, $\lfloor 600/(3\cdot 7)\rfloor=28$, and $\lfloor 600/(5\cdot 7)\rfloor=17$, as this is the number of elements that are divisible by 3 and 5, 3 and 7, or 5 and 7, respectively. Finally, we need to add the numbers of elements that are divisible by 3, 5, and 7, as these have been added three times and subtracted three times so far: $\lfloor 600/(3\cdot5\cdot7)\rfloor = 5$. In total we get $200+120+85-40-28-17+5 = 325$. Thus there are $600-325 = 275$ elements that are *not* divisible by 3, 5, or 7.

$\boxed{3}$ Basis step: $n=1$ gives $4\cdot1\cdot3\cdot5 = 1\cdot2\cdot5\cdot6$, which is true. Inductive step: Assuming that the equation holds for $n=k$, we show that it holds for $n=k+1$:

$$\begin{aligned}
4\sum_{i=1}^{k} i(i+2)(i+4) &= k(k+1)(k+4)(k+5)\\
\implies 4(k+1)(k+3)(k+5) + 4\sum_{i=1}^{k} i(i+2)(i+4) &= 4(k+1)(k+3)(k+5) + k(k+1)(k+4)(k+5)\\
\implies 4\sum_{i=1}^{k+1} i(i+2)(i+4) &= (k+1)(k+5)(4(k+3)+k(k+4))\\
&= (k+1)(k+5)(k+2)(k+6)\\
&= (k+1)((k+1)+1)((k+1)+4)((k+1)+5)
\end{aligned}$$

$\boxed{4}$ **4.1.12a)** $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta$, by the sine and cosine double-angle formulas.

b) Basis step: The equation clearly holds for $n=1$.
Inductive step: Assuming that the equation holds for $n = k \geq 1$, we prove it for $n = k+1$.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^k &= \cos k\theta + i\sin k\theta\\
\implies (\cos\theta + i\sin\theta)^{k+1} &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\
&= \cos k\theta\cos\theta + i\sin k\theta\cos\theta + i\cos k\theta\sin\theta - \sin k\theta\sin\theta\\
&= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\
&= \cos(k+1)\theta + i\sin(k+1)\theta
\end{aligned}$$

Again, we have used the trigonometric angle addition formulas.

c)

$$\begin{aligned}
\sqrt{2}(\cos 45^\circ + i\sin 45^\circ) &= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\\
&= 1+i\\
\implies (1+i)^{100} &= (\sqrt{2}(\cos 45^\circ + i\sin 45^\circ))^{100}\\
&= \sqrt{2}^{100}(\cos(100\cdot45)^\circ + i\sin(100\cdot45)^\circ)\\
&= 2^{50}(\cos 180^\circ + i\sin 180^\circ)\\
&= 2^{50}(-1)\\
&= -2^{50}
\end{aligned}$$

$\boxed{5}$ **4.1.16a)**

$$\begin{aligned}
s_3 &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{6} \\
&= 1 + 1 + \frac{2}{3} + \frac{2}{6} \\
&= 3
\end{aligned}$$

b)

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} + \frac{1}{2} = 2 \\
s_4 &= s_3 + s_3 \cdot \frac{1}{4} + \frac{1}{4} = 3 + \frac{3}{4} + \frac{1}{4} = 4
\end{aligned}$$

In the calculation of s_4, we have made the observation that any nomempty subset A of X_4 is of one of three types: Either $A \subseteq X_3$, or $A = B \cup \{4\}$, where B is a nonempty subset of X_3, or $A = \{4\}$. The contributions to the sum by the subsets of the first type add up to s_3. The subsets of the second type contribute with the exact same, only with an extra 4 in the denominator, so they add up to $s_4/4$. Finally, we get $1/4$ from $A = 4$.

c) In the cases we have checked so far, $s_n = n$, where $s_n = \sum_{\emptyset \neq A \subseteq X_n} 1/p_A$ and $X_n = \{1, 2, \ldots, n\}$. We conjecture that this holds for all $n \geq 1$. (It also holds for $n = 0$, as $X_0 = \emptyset$, meaning that s_0 is equal to the empty sum, which is 0.) For the basis step, we check that $s_1 = \sum_{\emptyset \neq A \subseteq \{1\}} 1/p_A = 1/1 = 1$. In the inductive step, we assume that $s_k = k$ for some $k \geq 1$. We can use the same reasoning as when we calculated s_4 above to see that $s_{k+1} = s_k + s_k \cdot 1/(k+1) + 1/(k+1) = k + k/(k+1) + 1/(k+1) = k + 1$, and we have shown the statement for $k+1$, which concludes the inductive step.

$\boxed{6}$ **4.1.17a)**

$$\begin{aligned}
H_{2^n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \cdots + 2^{n-1}\frac{1}{2^n} \\
&= \sum_{k=1}^{n} 2^{k-1}\frac{1}{2^k} \\
&= \sum_{k=1}^{n} \frac{1}{2} \\
&= \frac{n}{2}
\end{aligned}$$

b) In the basis step, we set $n = 1$. Then the equation says $H_1 = (2/2)H_2 - 2/4 = (1 + 1/2) - 1/2 = 1$, which is true.

Assume that the equation holds for $n = k$.

$$
\begin{aligned}
\sum_{j=1}^{k} jH_j &= \frac{k(k+1)}{2}H_{k+1} - \frac{k(k+1)}{4} \\
\implies \sum_{j=1}^{k+1} jH_j &= \frac{k(k+1)}{2}H_{k+1} - \frac{k(k+1)}{4} + (k+1)H_{k+1} \\
&= \frac{2(k+1)+k(k+1)}{2}H_{k+1} - \frac{k(k+1)}{4} \\
&= \frac{(k+1)(k+2)}{2}H_{k+1} - \frac{k(k+1)}{4} \\
&= \frac{(k+1)(k+2)}{2}\left(H_{k+2} - \frac{1}{k+2}\right) - \frac{k(k+1)}{4} \\
&= \frac{(k+1)(k+2)}{2}H_{k+2} - \frac{k+1}{2} - \frac{k(k+1)}{4} \\
&= \frac{(k+1)(k+2)}{2}H_{k+2} - \frac{(k+1)(k+2)}{4}
\end{aligned}
$$

We see that the equation then also holds for $n = k + 1$, and the induction step is done.

$\boxed{7}$ **(1)** We have $A \subseteq A \cup B$ and $A \cap B \subseteq B$. This gives $A \subseteq A \cup B \subseteq A \cap B \subseteq B$, and similarly we have $B \subseteq A$. Together, these inclusions imply $A = B$.

(2)

$$
\begin{aligned}
\overline{A \cap B} &= \{x | x \notin A \cap B\} \\
&= \{x | \neg(x \in A \wedge x \in B)\} \\
&= \{x | x \notin A \vee x \notin B)\} \\
&= \{x | x \notin A\} \cup \{x | x \notin B\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

(3) Suppose $x \in A \cap (B \cup C)$. Then x is in both A and either B or C. In the first case, $x \in A \cap B$, and in the second, $x \in A \cap C$. In either case, $x \in (A \cap B) \cup (A \cap C)$. Suppose instead $x \in (A \cap B) \cup (A \cap C)$. Then either $x \in A \cap B$ or $x \in A \cap C$. In either case, we have $x \in A$ and $x \in B \cup C$, so $x \in A \cap (B \cup C)$. Thus the two sets are contained in each other and must therefore be equal.

$\boxed{8}$ **4.1.2a)** Basis step: The statement for $n = 1$ is $2^0 = 2^1 - 1$, which is true.

Induction step:

$$\begin{aligned}
&\sum_{i=1}^{k} 2^{i-1} = 2^k - 1 \\
\implies &\sum_{i=1}^{k+1} 2^{i-1} = 2^k - 1 + 2^k \\
&\qquad\quad = 2 \cdot 2^k - 1 \\
&\qquad\quad = 2^{k+1} - 1
\end{aligned}$$

b) Basis step: The statement for $n = 1$ is $1 \cdot 2^1 = 2 + (1-1)2^{1+1}$, which is true.

Induction step:

$$\begin{aligned}
&\sum_{i=1}^{k} i2^{i} = 2 + (k-1)2^{k+1} \\
\implies &\sum_{i=1}^{k+1} i2^{i} = 2 + (k-1)2^{k+1} + (k+1)2^{k+1} \\
&\qquad\quad = 2 + 2k2^{k+1} \\
&\qquad\quad = 2 + ((k+1)-1)2^{(k+1)+1}
\end{aligned}$$

c) Basis step: The statement for $n = 1$ is $1 \cdot 1! = (1+1)! - 1$, which is true.

Induction step:

$$\begin{aligned}
&\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1 \\
\implies &\sum_{i=1}^{k+1} i \cdot i! = (k+1)! - 1 + (k+1)(k+1)! \\
&\qquad\quad = (1+k+1)(k+1)! - 1 \\
&\qquad\quad = (k+2)! - 1
\end{aligned}$$

$\boxed{9}$ Basis step: The statement for $n = 1$ is $1^3 = (1 \cdot (1+1)/2)^2$, which is true.

Induction step:

$$\begin{aligned}
\sum_{i=1}^{k} i^3 &= \left(\frac{k(k+1)}{2}\right)^2 \\
\implies \sum_{i=1}^{k+1} i^3 &= \frac{k^2(k+1)^2}{2^2} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{2^2} \\
&= \frac{(k^2+4k+4)(k+1)^2}{2^2} \\
&= \frac{(k+2)^2(k+1)^2}{2^2} \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2
\end{aligned}$$

$\boxed{10}$ Basis step: We see that $1^3 + 3 \cdot 1^2 + 2 \cdot 1 = 6$ is divisible by 6.

Induction step: Suppose that $k^3 + 3k^2 + 2k$ is divisible by 6. We have

$$\begin{aligned}
(k+1)^3 + 3(k+1)^2 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 + 2k + 2 \\
&= (k^3 + 3k^2 + 2k) + 3k^2 + 9k + 6 \\
&= (k^3 + 3k^2 + 2k) + 3k(k+3) + 6
\end{aligned}$$

Now, $k^3 + 3k^2 + 2k$ is divisible by 6 by the induction hypothesis, and 6 is of course divisible by 6. That leaves $3k(k+3)$. This is divisible by 3, but also by 2 since either k or $k+3$ must be an even number. Thus the whole expression is divisible by 3 and 2 and therefore 6, and we are done. (Note: one can also prove the statement without using induction by factorizing $n^3 + 3n^2 + 2n = n(n+1)(n+2)$ and observing that three consecutive integers must contain a number divisible by 2 and one divisible by 3.)

$\boxed{11}$ **4.1.19)** Suppose that $S(k)$ is true. We get

$$\begin{aligned}
\sum_{i=1}^{k} i &= \frac{(k+(1/2))^2}{2} \\
\implies \sum_{i=1}^{k+1} i^2 &= \frac{(k+(1/2))^2}{2} + k + 1 \\
&= \frac{k^2 + k + 1/4 + 2(k+1)}{2} \\
&= \frac{k^2 + 3k + 9/4}{2} \\
&= \frac{((k+1)+(1/2))^2}{2}
\end{aligned}$$

The left side of the equation is an integer is always an integer, while the right side is never an integer. This means that the equation is false for all n. This example shows that the basis step is necessary for an inductive proof to be valid.

$\boxed{12}$ **4.1.27)** Let T_i be the statement that $S(n_0), \ldots, S(n_1 + i)$ are all true. Then the assumption in a) in Theorem 2 is that T_0 is true and in b) that T_{k-n_1} implies T_{k-n_1+1} for all $k - n_1 \geq 0$. By Theorem 1, this implies that T_n holds for all $n \geq 0$. But since we have already assumed that $S(n_0), \ldots, S(n_1)$ all hold in a), this means that S_n holds for all $n \geq n_0$. Thus we have proved Theorem 2.

$\boxed{13}$ Note that all multiples of 6 are even, so $D = \{n \in \mathbb{N} \mid n \text{ is a multiple of } 6\}$. All multiples of 12 are multiples of 6 (an element of C can be written $12k = 6(2k)$), so $C \subseteq D$. But $6 \in D - C$, so $D \nsubseteq C$ and $C \neq D$.

$\boxed{14}$ **i)** We have $\neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b) \leftrightarrow (b \to \neg a)$. Thus

$$(\neg(a \wedge b) \wedge (\neg c \to b)) \leftrightarrow ((\neg c \to b) \wedge (b \to \neg a)) \to (\neg c \to \neg a) \leftrightarrow (a \to c).$$

ii) We assume that

1. $\neg(\neg p \vee q)$
2. $\neg z \to \neg s$
3. $(p \wedge \neg q) \to s$
4. $\neg z \vee r$

are all true, and want to prove r.

$1 \leftrightarrow p \wedge \neg q$, which together with 3 shows that s is true. 2 is equivalent to $s \to z$, and 4 to $z \to r$. Altogether we have s and $s \to z \to r$, which gives us r.