LØSNINGER LØRDAGSPRØVEN 2.12.06

1) $\int x\cos(x^2)\,dx = \frac{1}{2}\int 2x\cos(x^2)\,dx \underset{(u=x^2)}{=} \underline{\frac{1}{2}\sin(x^2)+C}$

2) $\int \underbrace{\ln(x^2+1)}_{u}\cdot \underbrace{1}_{v'}\,dx = x\ln(x^2+1) - \int \frac{x\cdot 2x}{x^2+1}\,dx$

$= x\ln(x^2+1) - \int\left(2 - \frac{2}{x^2+1}\right)dx$

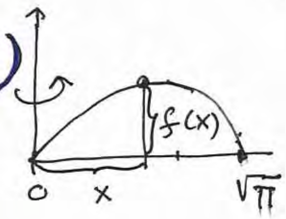
$= \underline{x\ln(x^2+1) - 2x + 2\arctan x + C}$

3) $u = \arcsin x \Rightarrow \sin u = x \Rightarrow \cos u\,du = dx$

$x = 0 \Rightarrow u = 0\,,\quad x = 1/2 \Rightarrow u = \frac{\pi}{6}$

Altså $\int_0^{1/2} e^{\arcsin x}\,dx = \underline{\int_0^{\pi/6} \cos u\, e^u\,du}$

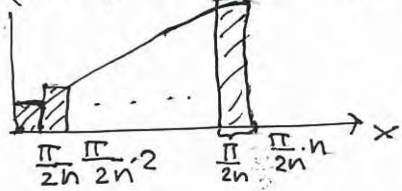
4) $F(x) = \int_1^{x^2} \frac{\sin t}{t}\,dt \Rightarrow F'(x) = \frac{\sin(x^2)}{x^2}\cdot D(x^2) = \underline{\frac{2\sin(x^2)}{x}}$

5) 

$V = \int_0^{\sqrt{\pi}} 2\pi x\sin(x^2)\,dx \underset{(u=x^2)}{=} -\pi\Big[\cos(x^2)\Big]_0^{\sqrt{\pi}} = \underline{2\pi}$

6) $\lim_{n\to\infty} \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\left(\frac{\pi}{2n}\cdot 2\right) + \sin\left(\frac{\pi}{2n}\cdot 3\right) + \cdots + \sin\left(\frac{\pi}{2n}\cdot n\right)\right]$

$= \int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x = \underline{1}$



7) $r^2 - 6r + 9 = (r-3)^2 = 0$ slik at $\underline{y = Ce^{3x} + Dxe^{3x}}$

$\underline{y' = 3Ce^{3x} + D(1+3x)e^{3x}}$. Setter inn startkravene:

$y(0) = 0 \Leftrightarrow Ce^0 = 0 \Leftrightarrow \underline{C=0}$, $y'(0) = 3 \Leftrightarrow \underline{3C + D = 3}$ ⊃: $\underline{D=3}$

Altså $\underline{y = 3xe^{3x}}$

8) $\int \frac{dy}{y-1} = \int \frac{dx}{1+x^2}$; $\ln|y-1| = \arctan x + C$,

$|y-1| = e^C e^{\arctan x}$; $y = 1 + Ke^{\arctan x}$

$y(0) = 0 \Leftrightarrow K = -1$ slik at $\underline{y = 1 - e^{\arctan x}}$

Alternativt løser du diff. lign. som 1. ordens lineær!

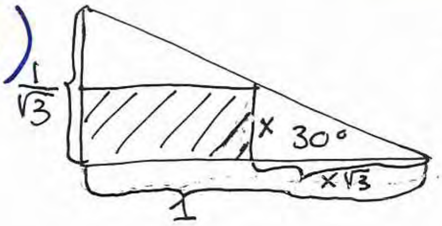
I) $$\frac{4x^2-x+5}{(x+1)(x^2-2x+2)} = \frac{A(x^2-2x+2)+(Bx+C)(x+1)}{(x+1)(x^2-2x+2)}$$

Oppfylt dersom
i) $A+B=4 \Leftrightarrow \underline{B=4-A}$
ii) $-2A+B+C=-1$
iii) $2A+C=5 \Leftrightarrow C=\underline{5-2A}$

i) og iii) innsatt i ii) gir $-5A=-10$ dvs. $\underline{A=2,\ B=2\ C=1}$

"smugler inn" "gjør godt igjen"

$$\int \frac{2x+1}{x^2-2x+2}\,dx = \int \frac{(2x-2)+2+1}{x^2-2x+2}\,dx = \ln(x^2-2x+2)+\int\frac{3\,dx}{(x-1)^2+1}$$
$$= \underline{\ln(x^2-2x+2)+3\arctan(x+1)+C}$$

II) 

$\frac{1}{\sqrt{3}} = \tan 30°$

$A=(1-\sqrt{3}x)x \quad , \quad 0 \le x \le \frac{1}{\sqrt{3}}$

$A'(x)=1-2\sqrt{3}x = 0 \Leftrightarrow x=\frac{1}{2\sqrt{3}}$

$A\left(\frac{1}{2\sqrt{3}}\right) = \underline{\frac{1}{4\sqrt{3}} = A_{max}}$: A kont. på $\left[0,\frac{1}{\sqrt{3}}\right]$, $A(0)=A\left(\frac{1}{\sqrt{3}}\right)=0$.

III) Har $|f(x)-f(y)| \le K|x-y|$ alle $x,y\in I$.
Da er f kontinuerlig i alle $a\in I$: For $\varepsilon>0$ gitt, så velger vi $|x-a|<\underline{\varepsilon/K=\delta}$, og $|f(x)-f(a)|<\frac{\varepsilon K}{K}=\varepsilon$. □

Bevis for <u>påstanden</u>: Da g er deriverbar, har vi for $x,y\in I$, $x<y$:

$$\frac{g(y)-g(x)}{y-x} = g'(c) \text{ der } c\in(x,y)$$ MIDDELVERDISETN (SEKANTSETN.)

Tilsvarende for $y<x$ slik at

$$\left|\frac{g(y)-g(x)}{y-x}\right| = |g'(c)| \text{ der } c \text{ mellom } x \text{ og } y$$

Siden g' er kontinuerlig på det lukkede, begrensede intervallet I er g' begrenset, dvs. $|g'(t)|\le K, t\in I$.
Altså (*) $|g(x)-g(y)| \le K|x-y|$ når $x,y\in I, x\ne y$.
Men (*) gjelder trivielt når $x=y$ $(0\le K\cdot 0)$.

□