Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **MA1101/MA6101 Basic Calculus I**

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Examination date: Dec. 11, 2018

Examination time (from–to): 9 a.m.–1 p.m.

Permitted examination support material: D: No printed or hand-written support material is allowed. A specific basic calculator is allowed. (Casio fx-82ES PLUS, Casio fx-82EX, Citizen SR-270X, Citizen SR-270X College, Hewlett Packard HP30S).

Other information:
The relative weighting (points) is given in each problem. It is advised to read through the exam before starting solving it; the perceived difficulty might not be in order of appearance. Solutions should be given in clear and unambiguous writing, with justifications for calculations and proofs (does not apply to the first problem). Drawings and sketchs might help communicating your solutions. Please ask in case of any uncertainties.

Language: English

Number of pages: 3

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 *(20p)*

Which of the following propositions are correct? Please mark your answers on a separate sheet with 'True', 'False', and 'Undecidable'. Justifications are not needed.

(i) The function cos is even.

(ii) $\lim_{x\to\infty} \frac{\ln(x^2)}{x} = 0$.

(iii) The function $x \mapsto x^3$ is everywhere strictly increasing.

(iv) $\lim_{n\to\infty} \frac{\pi}{n} \sum_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right) = \int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x$.

(v) A given continuous function satisfying $f(0) = 0$ is differentiable at the origin.

(vi) The differential equation $y' + x^2 y = 0$ is separable.

(vii) $\ln(1+x) = x - \mathcal{O}(x^2)$ for x close to 0.

(viii) Every bounded function has a global maximum.

(ix) If $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, then f is differentiable at x_0.

(x) Every continuously differentiable (C^1) function is also uniformly continuous.

Problem 2 *(10p)*

Integrate the following expressions.

(i) $\int \frac{\mathrm{d}x}{x(x-1)}$

(ii) $\int_{-\pi}^{\pi} \sin(x)\cosh(x)\,\mathrm{d}x$ $\qquad (\cosh(x) = \frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2})$

(iii) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x^2}$

Problem 3 *(10p)*

The function $\sinh\colon \mathbb{R} \to \mathbb{R}$ given by

$$\sinh(x) = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2},$$

can be written in the form $\sinh(x) = \sum_{k=0}^{\infty} a_k x^k$.

(i) Determine the coefficients a_k, for all $k = 0, 1, 2, \ldots$

(ii) Calculate the limit
$$\lim_{x \to 0} \frac{\sin(x) - \sinh(x)}{x^3}.$$

Problem 4 *(10p)*

For each of the following functions, determine whether they are (i) surjective (onto), (ii) injective (one-to-one), and (iii) invertible, on the sets given.

(i) $x \mapsto x + x^3 \colon \mathbb{R} \to \mathbb{R}$

(ii) $x \mapsto \cos(x) \colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0, 1]$

Problem 5 *(10p)*

(i) Find all critical (stationary) points to the function $f\colon \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \frac{\ln(1 + x^2)}{1 + x^2}.$$

(ii) Locate any global maxima and minima of f, and determine where the function is increasing/decreasing. Find any asymptotes as $x \to \pm\infty$.

Problem 6 *(10p)*

Simplify

$$\sum_{k=0}^{n} x^k, \qquad n \in \mathbb{N}.$$

For which values of x does $\lim_{n\to\infty} \sum_{k=0}^{n} x^k$ converge?

Problem 7 *(10p)*

The function $\sin\colon \mathbb{R} \to [-1, 1]$ is smooth (C^∞) and therefore continuous. Show that is also uniformly continuous.

Problem 8 *(10p)*

Consider, for $x \geq 0$, the initial-value problem

$$y'(x) - 6x\, y(x) = 0, \qquad y(0) = 1. \tag{$*$}$$

(i) The solution $x \mapsto y(x)$ of $(*)$ admits an inverse $y \mapsto x(y)$, $y \geq 1$. Formulate the initial-value problem corresponding to $(*)$, but for the inverse function $x = x(y)$.

(ii) Determine the solution y and its inverse.

Problem 9 *(10p)*

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$\sup_{x\in\mathbb{R}} |f(x)| \leq B,$$

for a finite number B. Show, using an ε/δ type of argument, that the function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_0^x f(s)\,\mathrm{d}s$$

is continuous at each $x_0 \in \mathbb{R}$.

A proof without ε/δ can at most give half of the points for this problem.