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1 Bestem hvilke av utsagnende som er sanne og usanne. Svar med «Sann» eller «Usann». *Det trengs ikke begrunnelse i denne oppgaven.*

a) Dersom $f : \mathbb{R} \to \mathbb{R}$ er en deriverbar funksjon slik at $f'(x)$ er begrenset. Da er $f(x)$ også begrenset.

b) Gitt ein begrenset funksjon $f : (0,1) \to \mathbb{R}$, finnes et punkt $x \in (0,1)$ slik at $f(x) > f(y)$ for alle $y \neq x$.

c) En deriverbar funksjon $f : \mathbb{R} \to \mathbb{R}$ som tilfredstiller $f'(x) < 0$ for alle $x \in \mathbb{R}$ er injektiv (1-1).

d) Dersom $f : \mathbb{R} \to \mathbb{R}$ er en begrenset og deriverbar funksjon, er $f'(x)$ også en begrenset funksjon.

e) Dersom $(a_n)_{n\in\mathbb{N}}$ er en begrenset følge i $\mathbb{R}$, så finnes det en delfølge $(b_k)_{k\in\mathbb{N}} = (a_{n_k})_{k\in\mathbb{N}}$ slik at $\lim_{n\to\infty} |b_k - b_{k+1}| = 0$.

f) Det finnes reelle tall $a, b \neq 0$ slik at $\int_{-1}^{1} (a-b)(\cos(x) + \sin(x))\, dx = -1$.

g) Enhver kontinuerlig funksjon $f : [0,1] \to \mathbb{R}$ er begrenset.

h) Gitt en kontiunuerlig funksjon $f : [0,1] \to \mathbb{R}$ slik at $f(0) = -2$, og $f(1) = 2$, finnes et punkt $x \in (0,1)$ slik at $f(x) = x$.

i) La $f : \mathbb{R} \to \mathbb{R}$ være kontiunuerlig i x. Da eksisterer grensen $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

j) Funksjonen

$$f = \begin{cases} x\sin(1/x) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

er kontiunuerlig på hele $\mathbb{R}$.

Solution: a) Usann; Moteksempel $f(x) = x$.
b)Usann; $\sup \neq \max$, moteksempel $f(x) = x$.
c)Sann; Funksjonen er strengt avtagende.
d)Usann; Moteksempel $f(x) = \sin(x^2)$.
e)Sann; Bolzano-Weierstraß teorem.
f)Sann; $\sin(x)$ er odde så vi kan velge a, b slik at $a - b = \dfrac{-1}{\int_{-1}^{1} \cos(x)\, dx}$.
g)Sann; Ekstremalverdisetnignen.
h)Sann; Skjæringssetningen på $g(x) = f(x) - x$.
i)Usann; Kontinuerlig $\not\Rightarrow$ deriverbar, moteksempel $f(x) = |x|$.
j)Sann; Vises f.eks. ved skviseteoremet.

2 La

$$f(x) = \arctan(x^2), \qquad x \in \mathbb{R},$$

hvor $\arctan(x)$ betegner den inverse funksjonen til $\tan(x)$. Bestem intervallene der f er voksende eller avtakende. Bestem også alle vertikale, horisontale, og skrå asymptoter til f. Lag en skisse av grafen til $y = f(x)$ med hjelp av dine svar. Finn eventuelle maksimums og minimumspunkter til $f(x)$ og tilsvarende maksimums og minimumsverdier.

Hint: Husk at

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1}\,.$$

Solution: We use the hint and calculate the derivative

$$f'(x) = \frac{2x}{x^4+1}.$$

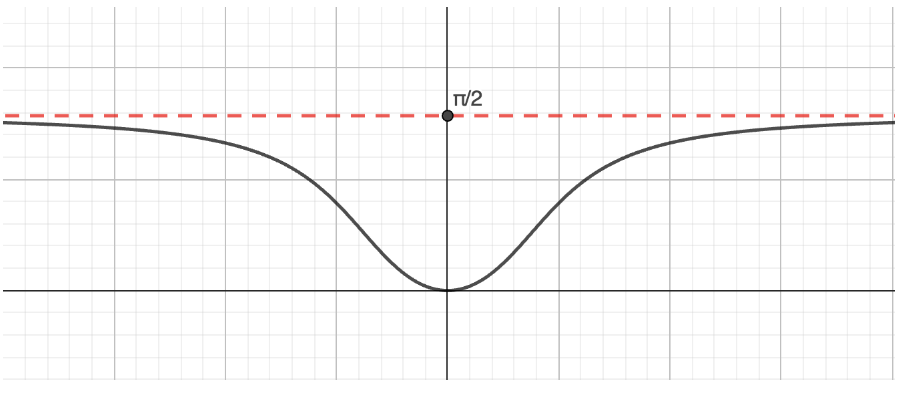
Note that the denominator $x^4 + 1$ is strictly positive. We therefore get that $f(x)$ is decreasing for $x \in (-\infty, 0)$, has a critical point when $x = 0$ and is increasing for $x \in (0, \infty)$. Alternatively, this conclusion is clear by noting that f is the composition of the strictly increasing function arctan with $x \mapsto x^2$.

Since $f(x)$ is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, 0)$, the smallest value $f(x)$ attains is $f(0) = 0$.

We make some observations for determining the asymptotes. We note that $f(x) = f(-x)$, i.e. the function is even, $f(x)$ is continuous on all of $\mathbb{R}$ as it is the composition of two continuous functions and $f(x)$ is bounded. Continuity and boundedness tells us that there can be no vertical or slant asymptotes. Since $\lim_{x\to\frac{\pi}{2}} \tan(x) = \infty$, we get that

$$\lim_{x\to\infty} \tan(x^2) = \lim_{x\to\infty} \tan(x) = \frac{\pi}{2}.$$

By the even symmetry we get the horizontal asymptote $y = \frac{\pi}{2}$ in both the negative and positive directions. Example sketch below.



3 **a)** Beregn Taylorpolynomet av grad 4 rundt punktet $x = 0$ til funksjonen

$$f(x) = \sin(x)e^x.$$

b) Ved å bruke Taylorpolynomet av grad 3 rundt punktet $x = 0$, finn en tilnærming til $\sin(1/2)\sqrt{e}$ med feil mindre enn $0.03125 = (\frac{1}{2})^5$.

Hint: Husk at

$$f(x) = T_n(x) + R_n(x), \qquad R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Solution: a): We start by calculating the 4 first derivatives of $f(x)$.

$$\begin{aligned}
f'(x) &= \sin(x)e^x + \cos(x)e^x \\
f''(x) &= (\sin(x)e^x + \cos(x)e^x) + (\cos(x)e^x - \sin(x)e^x) = 2\cos(x)e^x \\
f'''(x) &= 2(\cos(x)e^x - \sin(x)e^x) \\
f''''(x) &= 2((\cos(x)e^x - \sin(x)e^x) - (\sin(x)e^x + \cos(x)e^x)) = -4\sin(x)e^x
\end{aligned}$$

Since $e^0 = 1, \sin(0) = 0, \cos(0) = 1$ we get

$$\begin{aligned}
T_4(x) &= 0 + x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 \\
&= x + x^2 + \frac{1}{3}x^3.
\end{aligned}$$

b) Using the hint as well as the rough estimate $e^c \leq e$, $c \leq 1$ and $e < 3$ we get

$$|R_4(\frac{1}{2})| = \frac{|-4\sin(c)e^c|}{4!}\frac{1}{2^4} < \frac{e}{3!}\frac{1}{2^4} < \frac{1}{2^5}.$$

So we can bound the error by

$$|f(\frac{1}{2}) - T_3(\frac{1}{2})| = |\sin(\frac{1}{2})\sqrt{e} - T_3(\frac{1}{2})| < \frac{1}{2^5}.$$

Our estimate is then

$$\sin(\frac{1}{2})\sqrt{e} \approx 0.5 + 0.5^2 + \frac{0.5^3}{3} = 0.791666\ldots.$$

4 **a)** Beregn $\lim_{x\to 0} \frac{x^{2024} - x^2 + 3x}{x^{2023} + x}$.

b) Beregn $\lim_{x\to\infty} x^2 \sin\frac{1}{x^2}$.

c) Vis at

$$\lim_{x\to\infty} \frac{1}{x}\cos\left(\frac{1}{x}\right) = 0.$$

Solution:
a) All the conditions for using l'Hôpital's rule are satisified so we get

$$\lim_{x\to 0} \frac{x^{2024} - x^2 + 3x}{x^{2023} + x} = \lim_{x\to 0} \frac{2024x^{2023} - 2x + 3}{2023x^{2022} + 1} = 3.$$

b)

$$\lim_{x\to\infty} x^2 \sin\frac{1}{x^2} = \lim_{x\to 0} \left(\frac{1}{x}\right)^2 \sin\left(\frac{1}{\frac{1}{x}^2}\right) = \lim_{x\to 0} \frac{1}{x} \sin\left(\frac{1}{\frac{1}{x}}\right) = \lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

The last limit can be computed using for example l'Hôpital's rule or Taylor series.

c) We know that $\cos\left(\frac{1}{x}\right)$ is bounded by $\left|\cos\left(\frac{1}{x}\right)\right| \leq 1$. Since $\frac{1}{x} \xrightarrow{x\to\infty} 0$ we get

$$\lim_{x\to\infty} \left|\frac{1}{x} \cos\left(\frac{1}{x}\right)\right| = 0 \implies \lim_{x\to\infty} \frac{1}{x} \cos\left(\frac{1}{x}\right) = 0.$$

5 Betrakt følgen $(a_n)_{n=1}^{\infty}$, rekursivt definiert av at

$$a_1 = \frac{1}{2}, \qquad a_{n+1} = \frac{1}{2 + \frac{1}{2+a_n}}, \qquad n \geq 1.$$

- **a)** Vis ved hjelp av induksjon at følgen er avtagende.
- **b)** Vis at følgen er begrenset.
- **c)** Motiver at følgen er konvergent og beregn $\lim_{n\to\infty} a_n$.

Solution:
a) We want to show $a_{n+1} < a_n$. We start by showing that our base case $n = 1$ holds:

$$a_2 = \frac{1}{2 + \frac{1}{2+\frac{1}{2}}} = \frac{1}{2 + \frac{1}{2.5}} < \frac{1}{2} = a_1.$$

Now consider the functions $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{2+\frac{1}{2+x}}$. Clearly f is increasing. For the induction step assume that $a_n > a_{n+1}$ for some $n \geq 1$. We have

$$a_n > a_{n+1} \implies a_{n+1} = f(a_n) \geq f(a_{n+1}) = a_{n+2}.$$

b) The upper bound $a_n \leq \frac{1}{2}$ is given to us from **a)**. For the lower bound we note that if $x > 0$ then $2 + x > 0 \implies \frac{1}{2+x} > 0$. So the sequence is bounded from below by 0.

c) We know that any bounded and decreasing sequence is convergent in $\mathbb{R}$, so the sequence converges. We now denote the limit of the sequence by L. Then

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2 + \frac{1}{2+a_n}} = \frac{1}{2 + \frac{1}{2+\lim_{n\to\infty} a_n}} = \frac{1}{2 + \frac{1}{2+L}}.$$

Solving for L we get two possible solutions $L = \sqrt{2} - 1$ and $L = -1 - \sqrt{2}$. But since the sequence is bounded from below by 0 we get $L = \sqrt{2} - 1$.

6 **a)** Beregn $\int_0^{\pi/4} x \sin x \, dx$.

b) Beregn $\int_0^4 \frac{x}{\sqrt{x+1}} \, dx$.

c) Beregn $\int_0^\infty e^{-x} \cos(e^{-x}) \, dx$.

Solution: a) Using integration by parts we get

$$\int_0^{\pi/4} x \sin x \, dx = \Big[- x \cos(x) \Big]_{x=0}^{x=\frac{\pi}{4}} + \int_0^{\pi/4} \cos x \, dx = \frac{1}{\sqrt{2}} - \frac{\pi}{4} \frac{1}{\sqrt{2}}$$

b) Using the substitution $u = x + 1$ we get

$$\int_0^4 \frac{x}{\sqrt{x+1}} \, dx = \int_1^5 \frac{u-1}{\sqrt{u}} \, du = \int_1^5 \sqrt{u} \, du + \int_1^5 \frac{-1}{\sqrt{u}} \, du.$$

Now using standard rules for integration we get

$$\int_1^5 \sqrt{u} \, du + \int_1^5 \frac{-1}{\sqrt{u}} \, du = 2/3 \Big[u^{\frac{3}{2}} \Big]_{u=1}^{u=5} - 2 \Big[\sqrt{u} \Big]_{u=1}^{u=5} = \sqrt{5}\frac{4}{3} + \frac{4}{3} = \frac{4}{3}(\sqrt{5} + 1)$$

c) Using the substitution $u = e^{-x}$ we get

$$\int_0^\infty e^{-x} \cos(e^{-x}) \, dx = \int_1^0 \cos(u) \, du = - \int_0^1 \cos(u) \, du = \sin(1) - \sin(0) = \sin(1)$$

7 Ved bruk av sammenligningstesten, vis at

a) $\int_1^\infty \frac{x}{x^3 + \ln x} \, dx$ er konvergent.

b) $\int_1^\infty \frac{x}{x^2 + \ln x} \, dx$ er divergent.

Solution: a) Note first that $\frac{\ln(x)}{x} > 0$ for every $x \in (1, \infty)$. We therefore have

$$\frac{x}{x^3 + \ln(x)} = \frac{1}{x^2 + \frac{\ln(x)}{x}} < \frac{1}{x^2} \implies \int_1^t \frac{x}{x^3 + \ln x} \, dx < \int_1^t \frac{1}{x^2} \, dx.$$

Then since

$$\int_1^\infty \frac{1}{x^2} \, dx \ \Big(= 1 \Big)$$

converges so will $\displaystyle\int_1^\infty \frac{x}{x^3+\ln x}\,dx$ by the comparison test.

b) We note that $\frac{\ln(x)}{x} < C$ for some constant C and $x \in [1,\infty)$. So

$$\frac{x}{x^2+\ln(x)} = \frac{1}{x+\frac{\ln(x)}{x}} > \frac{1}{x+C}\,.$$

Then for big enough t we have

$$\int_1^t \frac{x}{x^2+\ln(x)}\,dx > \int_1^t \frac{1}{x+C}\,dx = \int_{1+C}^{t+C} \frac{1}{x}\,dx.$$

So by comparison we get that

$$\int_1^\infty \frac{x}{x^2+\ln x}\,dx,$$

diverges since $\displaystyle\int_{1+C}^\infty \frac{1}{x}\,dx$ diverges.