MA1101 Grunnkurs i analyse 1

Løsningsforslag Øving 11

Høst 2024

Innleveringsfrist: Mandag 11. November
Lever øvingen i øvsys. Du velger selv om du leverer på norsk eller engelsk. Ved ønske om grundig retting, spesifiser oppgaver du ønsker det på i øvsys. Det viktigste er *hvordan* du løser oppgaven, ikke selve løsningen.

$\boxed{1}$ Evaluer de bestemte integralene.

a) $\displaystyle\int_0^4 \frac{x^3}{\sqrt{x^2+1}}\,\mathrm{d}x$ b) $\displaystyle\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\,\mathrm{d}x$

Løsning: Oppgave 1

a)

$$\int_0^4 x^3(x^2+1)^{-\frac{1}{2}}\,\mathrm{d}x.$$

Let $u = x^2+1$, $x^2 = u-1$, $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$\begin{aligned}\int_0^4 x^3(x^2+1)^{-\frac{1}{2}}\,\mathrm{d}x =& \frac{1}{2}\int_1^{17}(u-1)u^{-\frac{1}{2}}\,\mathrm{d}u \\ =& \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\Big|_1^{17} \\ =& \frac{17\sqrt{17}-1}{3} - (\sqrt{17}-1) = \frac{14\sqrt{17}}{3} + \frac{2}{3}.\end{aligned}$$

b)

$$\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\,\mathrm{d}x.$$

Let $u = \pi\ln(x)$, $\mathrm{d}u = \frac{\pi}{x}\,\mathrm{d}x$.

$$\begin{aligned}\int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x}\,\mathrm{d}x =& \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \sin(u)\,\mathrm{d}u \\ =& -\frac{1}{\pi}\cos(u)\Big|_0^{\frac{\pi}{2}} \\ =& -\frac{1}{\pi}(0-1) = \frac{1}{\pi}.\end{aligned}$$

$\boxed{2}$ Bruk de trigonometriske identitetene

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta) \quad \text{og} \quad \sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

til å evaluere følgende bestemte integral:

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos(x)}\,\mathrm{d}x \quad \text{og} \quad \int_0^{\frac{\pi}{2}} \sqrt{1-\sin(x)}\,\mathrm{d}x.$$

Løsning: Oppgave 2

For det første integralet ser vi at

$$\cos(x) = \cos(2 \cdot \tfrac{x}{2}) = 2\cos^2(\tfrac{x}{2}) - 1$$

Dermed kan vi skrive om uttrykket i rottegnet til en finere form:

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{1+\cos(x)}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\left(\frac{x}{2}\right)}\,\mathrm{d}x \\ &= \sqrt{2}\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\mathrm{d}x \\ &= 2\sqrt{2}\sin\left(\frac{x}{2}\right)\Big|_0^{\frac{\pi}{2}} = 2.\end{aligned}$$

Det andre integralet er kanskje ikke like åpenbart hvor man skal begynne, men vi har fått et hint om en start, nemlig bruke

$$\sin(x) = \cos(\tfrac{\pi}{2} - x).$$

Vi får dermed at vi skal finne

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos\left(\tfrac{\pi}{2} - x\right)}\,\mathrm{d}x$$

Vi aner en substitusjon i gjære, og den naturlige kandidat er $u = \frac{\pi}{2} - x$ som gir $\mathrm{d}u = -\mathrm{d}x$, og dermed

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos\left(\tfrac{\pi}{2} - x\right)}\,\mathrm{d}x = -\int_{-\frac{\pi}{2}}^{0} \sqrt{1-\cos(u)}\,\mathrm{d}u = \int_0^{\frac{\pi}{2}} \sqrt{1-\cos(u)}\,\mathrm{d}u.$$

Den siste likheten kommer av at $\sqrt{1-\cos(u)}$ er en odde funksjon. Nå kan vi la oss inspirere av hva vi gjorde i første integral, og benytte oss av

$$\cos(u) = \cos(2 \cdot \tfrac{u}{2}) = 1 - 2\sin^2(\tfrac{u}{2})$$

og dermed

$$\int_0^{\frac{\pi}{2}} \sqrt{1-\cos(u)}\mathrm{d}u = \int_0^{\frac{\pi}{2}} \sqrt{2\sin^2(\tfrac{u}{2})}\,\mathrm{d}u = \sqrt{2}\int_0^{\frac{\pi}{2}} \sin(\tfrac{u}{2})\,\mathrm{d}u$$

Vi kan enten regne ut integralet rett fra her, eller flotte oss med enda en substitusjon før vi er fornøyde. La $w = \frac{u}{2}$ og dermed $\mathrm{d}u = 2\mathrm{d}w$: som gir

$$\begin{aligned}\sqrt{2}\int_0^{\frac{\pi}{2}} \sin(\tfrac{u}{2})\,\mathrm{d}u &= 2\sqrt{2}\int_0^{\pi/4} \sin(w)\,\mathrm{d}w \\ &= 2\sqrt{2}\left[-\cos(w)\right]_0^{\pi/4} \\ &= 2\sqrt{2}\left(-\frac{\sqrt{2}}{2}+1\right) \\ &= 2\sqrt{2}-2\end{aligned}$$

$\boxed{3}$ Skisser og finn arealet av området i planet avgrenset av $2y = 4x - x^2$ og $2y + 3x = 6$.

Løsning: Oppgave 3

Sjekk **Definisjon 3.3.1** i boken for areal mellom grafer.

Kurvene $2y = 4x - x^2$ og $2y + 3x = 6$ kan sees som grafene til funksjonene

$$f(x) = -\tfrac{x^2}{2} + 2x$$

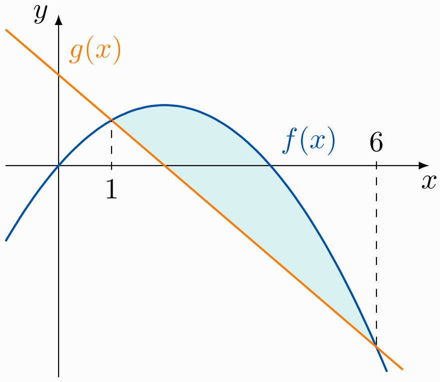
og

$$g(x) = -\tfrac{3}{2}x + 3.$$

Vi finner skjæringspunktene til kurvene:

$$\begin{aligned} f(x) &= g(x) \\ -\tfrac{x^2}{2} + 2x &= -\tfrac{3}{2}x + 3 \\ -x^2 + 4x &= -3x + 6 \\ 0 &= x^2 - 7x + 6 \\ 0 &= (x-1)(x-6)\end{aligned}$$

De skjærer hverandre altså for $x = 1$ og $x = 6$.



Vi har nå at siden $f(x) \geq g(x)$ for $x \in [1,6]$ vil arealet være gitt ved

$$\begin{aligned}\text{Areal} &= \int_1^6 f(x) - \int_1^6 g(x)\,\mathrm{d}x \\ &= \int_1^6 f(x) - g(x)\,\mathrm{d}x \\ &= \int_1^6 \left(-\frac{1}{2}x^2 + 2x + \frac{3}{2}x - 3\right)\,\mathrm{d}x \\ &= \int_1^6 \left(-\frac{1}{2}x^2 + \frac{7}{2}x - 3\right)\,\mathrm{d}x \\ &= \left[-\frac{1}{6}x^3 + \frac{7}{4}x^2 - 3x\right]_1^6 \\ &= \frac{125}{12} \text{ kvadratenheter.}\end{aligned}$$

$\boxed{4}$ Evaluer de ubestemte integralene under.

a) $\displaystyle\int x\ln(x)\,\mathrm{d}x$ b) $\displaystyle\int xe^{\sqrt{x}}\,\mathrm{d}x$ c) $\displaystyle\int \frac{x}{9x^2+6x+2}\,\mathrm{d}x$

Hint c): Finn a og b slik at $\frac{x}{9x^2+6x+2} = \frac{x}{(ax+b)^2+c}$, *og husk at den deriverte av arctan(x) er* $\frac{1}{1+x^2}$

Løsning: Oppgave 4

a) Let $u = \ln(x)$ and $v' = x$, then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$ and we get

$$\int x\ln(x)\,dx = \frac{1}{2}\ln(x)x^2 - \frac{1}{2}\int x\,dx = \frac{2\ln(x)x^2 - x^2}{4}.$$

b) Substitute $u = \sqrt{x}$, which implies $u^2 = x$. We then have that $2udu = dx$ which yields

$$\begin{aligned}\int xe^{\sqrt{x}}dx &= \int u^2e^u(2uds) \\ &= 2\int u^3e^u\,du\end{aligned}$$

To tame this beast, we will have to use integration by parts 3 times to work down the u^3 factor until it disappears.

$$\begin{aligned}\int u^3e^u\,du &= u^3e^u - 3\int u^2e^u\,du, \\ \int u^2e^u\,du &= u^2e^u - 2\int ue^u\,du, \\ \int ue^u\,du &= ue^u - \int e^u = ue^u - e^u \\ \implies 2\int u^3e^u\,du &= 2e^u\Big(u^3 - 3\big[u^2 - 2(u-1)\big]\Big) + C \\ &= e^u\big(2u^3 - 6u^2 + 12u - 12\big) + C.\end{aligned}$$

Plugging in $u = \sqrt{x}$ into this, we get

$$\int xe^{\sqrt{x}}\,dx = e^{\sqrt{x}}(2x^{3/2} - 6x + 12\sqrt{x} - 12) + C$$

c)

$$\int \frac{x}{9x^2+6x+2}\,\mathrm{d}x = \int \frac{x}{(3x+1)^2+1}\,\mathrm{d}x.$$

Let $u = 3x+1$, $\mathrm{d}u = 3\,\mathrm{d}x$. Then

$$\begin{aligned}\frac{1}{9}\int \frac{u-1}{u^2+1}\,\mathrm{d}u =& \frac{1}{9}\int \frac{u}{u^2+1}\,\mathrm{d}u - \frac{1}{9}\int \frac{1}{u^2+1}\,\mathrm{d}u \\ =& \frac{1}{18}\ln(u^2+1) - \frac{1}{9}\arctan u + C \\ =& \frac{1}{18}\ln(9x^2+6x+2) - \frac{1}{9}\arctan(3x+1) + C.\end{aligned}$$

$\boxed{5}$ Finn de trigonometriske integralene.

a) $\displaystyle\int \frac{(\cos x)^5}{(\sin x)^{3/2}}\,dx$ b) $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$

Løsning: Oppgave 5

a) Using that $\cos(x)^2 = 1 - \sin(x)^2$, we can write

$$\int \frac{(\cos x)^5}{(\sin x)^{3/2}}\,dx = \int \frac{(1-\sin(x)^2)^2\cos(x)}{(\sin x)^{3/2}}\,dx.$$

Now with the substitution $u = \sin(x)$, we get $du = \cos(x)dx$ and hence

$$\int \frac{(1-\sin(x)^2)^2\cos(x)}{(\sin x)^{3/2}}\,dx = \int \frac{(1-u^2)^2}{u^{3/2}}\,du.$$

Expanding it out and integrating it like we usually do polynomials, we get

$$\begin{aligned}\int \frac{(1-u^2)^2}{u^{3/2}}\,du &= \frac{2(3u^4 - 14u^2 - 21)}{21\sqrt{u}} + C \\ &= \frac{2(3\sin(x)^4 - 14\sin(x)^2 - 21)}{21\sqrt{\sin(x)}} + C.\end{aligned}$$

b) To solve the definite integral $\int_{-3}^{3}\sqrt{9-x^2}\,dx$, use the substitution $x = 3\sin(\theta)$, $dx = 3\cos(\theta)\,d\theta$. The limits of integration change to $\theta = -\frac{\pi}{2}$ when $x = -3$ and $\theta = \frac{\pi}{2}$ when $x = 3$.

The integral becomes:

$$9\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta$$

Using the trigonometric identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$, the integral simplifies to:

$$\frac{9}{2}\int_{-\pi/2}^{\pi/2} (1+\cos(2\theta))\, d\theta$$

Integrating, we get:

$$\frac{9}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\pi/2}^{\pi/2} = \frac{9\pi}{2}$$

$\boxed{6}$ **a)** Finn Taylorrekken av grad 3 rundt punktet $x = 0$ til funksjonen f gitt ved

$$f(x) = \int_0^x e^{\sin(x)}\, dx.$$

b) Finn en tilnærming til $f(0,1)$ med feil mindre enn $0,001$.

Løsning: Oppgave 6

a) We compute derivatives of the function as

$$\begin{aligned} f'(x) &= e^{\sin(x)} \implies f'(0) = 1, \\ f''(x) &= \cos(x)e^{\sin(x)} \implies f''(0) = 1, \\ f'''(x) &= (\cos(x)^2 - \sin(x))e^{\sin(x)} \implies f'''(0) = 1. \end{aligned}$$

Plugging this and $f(0) = 0$ into the third order Taylor polynomial we get

$$T_3(x) = x + \frac{x^2}{2} + \frac{x^3}{6}.$$

b) If we use the second order Taylor polynomial as our approximation, we can estimate the error using the third term. Specifically, the error is bounded by

$$|R_2(x)| = \left|\frac{f'''(c)}{3!}x^3\right| = \left|\frac{(\cos(c)^2 - \sin(c))e^{\sin(c)}}{6}x^3\right| \le \frac{2}{6}0.1^3 < 0.1^3 = 0.001$$

where $c \in (0, 0.1)$. The approximation is then

$$f(0,1) \approx 0.1 + \frac{0.1^2}{2} = 0.105.$$

$\boxed{7}$ Bestem om de uegentlige integralene konvergerer eller divergerer ved direkte utregning av grensene. Finn den eksakte verdien til grensen der den konvergerer.

a) $\displaystyle\int_0^\infty \frac{x}{e^x}\, dx.$

b) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt[3]{x^2}}\, dx.$

c) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx.$

Løsning: Oppgave 7

a) Using integration by parts we get

$$\begin{aligned}\int_0^T \frac{x}{e^x}\, dx &= \int_0^T x\,(-e^{-x})'\, dx \\ &= \big[-xe^{-x}\big]_0^T - \int_0^T -e^{-x}\, dx \\ &= -Te^{-T} - (e^{-T} - e^0) \\ &= -e^{-T}(T+1) + 1.\end{aligned}$$

So $\displaystyle\int_0^{\infty} \frac{x}{e^x}\, dx = \lim_{T\to\infty} 1 - e^{-T}(T+1) = 1.$

b) The antiderivative of $\dfrac{1}{\sqrt[3]{x^2}}$ is given by $\dfrac{3x}{\sqrt[3]{x^2}} = 3x^{\frac{1}{3}}$. So
$\displaystyle\int_1^{\infty} \frac{1}{\sqrt[3]{x^2}}\, dx = \lim_{T\to\infty} \int_1^T \frac{1}{\sqrt[3]{x^2}}\, dx = \lim_{T\to\infty} 3T^{\frac{1}{3}} - 1 \to \infty$. I.e. the integral diverges.

c) We use the substitution $x = \sin(t)$ which gives $dx = \cos(t)dt$. Then we get

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx &= \lim_{T\to 1^-} \int_{-T}^{T} \frac{1}{\sqrt{1-x^2}\, dx} \\ &= \lim_{T\to 1^-} \int_{\arcsin(-T)}^{\arcsin(T)} \frac{1}{\sqrt{1-\sin^2(t)}} \cos(t)\, dt \\ &= \lim_{T\to 1^-} \int_{\arcsin(-T)}^{\arcsin(T)} \frac{1}{\cos(t)} \cos(t)\, dt \\ &= \lim_{T\to 1^-} \int_{\arcsin(-T)}^{\arcsin(T)} 1\, dt \\ &= \lim_{T\to 1^-} \arcsin(T) - \arcsin(-T) \\ &= \frac{\pi}{2} - \frac{-\pi}{2} \\ &= \pi.\end{aligned}$$