MA1101 Grunnkurs i analyse 1

Løsningsforslag Øving 12

Høst 2023

Innleveringsfrist: Mandag 20. November
Lever øvingen i øvsys. Du velger selv om du leverer på norsk eller engelsk. Ved ønske om grundig retting, spesifiser oppgaver du ønsker det på i øvsys. Det viktigste er *hvordan* du løser oppgaven, ikke selve løsningen.

1 Bruk delvis integrasjon til å regne ut

a) $\displaystyle\int_0^1 x\cos(x)\,\mathrm{d}x,$ b) $\displaystyle\int_0^\pi \frac{\sin(x)}{e^x}\,\mathrm{d}x,$

Løsning: Oppgave 1

a)

$$\int_0^1 x\cos(x)\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}\Big(\sin(x)\Big) = x\sin(x)\Big|_0^1 - \int_0^1 \sin(x)\,\mathrm{d}x$$
$$= \sin(1) + \cos(x)\Big|_0^1 = \sin(1) + \cos(1) - 1.$$

b)

$$I = \int_0^\pi \sin(x)e^{-x}\,\mathrm{d}x = -\int_0^\pi e^{-x}\,\mathrm{d}\Big(\cos(x)\Big) = -\left[e^{-x}\cos(x)\Big|_0^\pi - \int_0^\pi \cos(x)\,\mathrm{d}(e^{-x})\right]$$
$$= -\left[-e^{-\pi} - 1 + \int_0^\pi \cos(x)e^{-x}\,\mathrm{d}x\right] = e^{-\pi} + 1 - \int_0^\pi e^{-x} d\Big(\sin(x)\Big)$$
$$= e^{-\pi} + 1 - \left[e^{-x}\sin(x)\Big|_0^\pi + \int_0^\pi \sin(x)e^{-x}\,\mathrm{d}x\right] = e^{-\pi} + 1 - I,$$

which gives

$$I = e^{-\pi} + 1 - I, \quad I = \frac{e^{-\pi}+1}{2}.$$

2 Regn ut integralene under

a) $\displaystyle\int \frac{1}{x^3 - 4x^2 + 3x}\,\mathrm{d}x$ b) $\displaystyle\int \frac{1}{e^{2x} - 4e^x + 4}\,\mathrm{d}x$

Hint for b): Bruk variabelskiftet $u = e^x$.

Løsning: Oppgave 2

a)

$$\begin{aligned}\frac{1}{x^3-4x^2+3x} &= \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3}\\ &= \frac{A(x^2-4x+3)+B(x^2-3x)+C(x^2-x)}{x^3-4x^2+3x}\end{aligned}$$

which gives the system of (linear) equations

$$\begin{cases} A+\ \ B+C=0\\ -4A-3B+C=0\\ 3A \qquad\qquad =1\end{cases}$$

and the consequent solutions:

$$A=\frac{1}{3},\quad B=-\frac{1}{2},\quad C=\frac{1}{6}.$$

Therefore, we have

$$\begin{aligned}\int\frac{1}{x^3-4x^2+3x}\,\mathrm{d}x &= \frac{1}{3}\int\frac{1}{x}\,\mathrm{d}x-\frac{1}{2}\int\frac{1}{x-1}\,\mathrm{d}x+\frac{1}{6}\int\frac{1}{x-3}\,\mathrm{d}x\\ &= \frac{1}{3}\ln|x|-\frac{1}{2}\ln|x-1|+\frac{1}{6}\ln|x-3|+C.\end{aligned}$$

b)

$$\begin{aligned}\int\frac{1}{e^{2x}-4e^x+4}\,\mathrm{d}x &= \int\frac{1}{(e^x-2)^2}\,\mathrm{d}x,\quad \text{let}\quad u=e^x,\quad \mathrm{d}u=e^x\,\mathrm{d}x\\ &= \int\frac{1}{u(u-2)^2}\,\mathrm{d}x.\end{aligned}$$

We now can use the method of partial fraction decomposition:

$$\begin{aligned}\frac{1}{u(u-2)^2} &= \frac{A}{u}+\frac{B}{u-2}+\frac{C}{(u-2)^2}\\ &= \frac{A(u^2-4u+4)+B(u^2-2u)+Cu}{u(u-2)^2}\end{aligned}$$

From this we obtain the following system of (linear) equations

$$\begin{cases} A+\ \ B \qquad =0\\ -4A-2B+C=0\\ 4A \qquad\qquad =1\end{cases}$$

which has solutions

$$A=\frac{1}{4},\quad B=-\frac{1}{4},\quad C=\frac{1}{2}$$

Thus,

$$\begin{aligned}\int \frac{1}{u(u-2)^2}\,\mathrm{d}u =& \frac{1}{4}\int \frac{1}{u}\,\mathrm{d}u - \frac{1}{4}\int \frac{1}{u-2}\,\mathrm{d}u + \frac{1}{2}\int \frac{1}{(u-2)^2}\,\mathrm{d}u \\ =& \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2| - \frac{1}{2}\frac{1}{(u-2)} + C \\ =& \frac{x}{4} - \frac{1}{4}\ln|e^x-2| - \frac{1}{2(e^x-2)} + C.\end{aligned}$$

3 **a)** Vis at

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x.$$

Hint: Ta logaritmen.

Beregn så grenseverdiene

b)

$$\lim_{n\to\infty}\left(1+\frac{1}{xn}\right)^n, \qquad x \neq 0$$

c)

$$\lim_{n\to\infty}\left(1+2e^{-n}\right)^{e^n}$$

Løsning: Oppgave 3

a)

If $x=0$, there is nothing to prove; both sides of the identity are 1. If $x \neq 0$, let $h = \frac{x}{n}$. As n tends to infinity, h approaches 0. Thus,

$$\begin{aligned}\lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty} n\ln\left(1+\frac{x}{n}\right) \\ &= \lim_{n\to\infty} x\frac{\ln\left(1+\frac{x}{n}\right)}{\frac{x}{n}} \\ &= x\lim_{h\to 0}\frac{\ln(1+h)}{h} \quad \text{(where } h = \frac{x}{n}\text{)} \\ &= x\lim_{h\to 0}\frac{\ln(1+h)-\ln 1}{h} \quad \text{(since } \ln 1 = 0\text{)} \\ &= x\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln(t)\right)\Big|_{t=1} \quad \text{(by the definition of derivative)} \\ &= x\cdot\frac{1}{t}\Big|_{t=1} = x.\end{aligned}$$

Since ln is differentiable, it is continuous. Hence, changing the order of ln and limit, we have

$$\ln\left(\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\right) = \lim_{n\to\infty}\ln\left(1+\frac{x}{n}\right)^n = x.$$

Taking exponentials of both sides gives the required formula.

b) Using **a)**, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{xn}\right)^n = \lim_{n\to\infty}\left[\left(1+\frac{1}{xn}\right)^{xn}\right]^{\frac{1}{x}} = e^{\frac{1}{x}}.$$

c) Using **a)**, we have

$$\lim_{n\to\infty}\left(1+2e^{-n}\right)^{e^n} = \lim_{n\to\infty}\left(1+\frac{2}{e^n}\right)^{e^n} = e^2.$$

4 Bruk sammenligningstesten for uegentlige integraler til å avgjøre hvorvidt følgende integraler konvergerer

a) $\displaystyle\int_e^\infty \frac{1-\cos(x)}{e^{x^2}}\,dx$

b) $\displaystyle\int_{10}^\infty \frac{1}{x-e^{-x}}\,dx$

c) $\displaystyle\int_0^1 \frac{1}{\arctan(x)}\,dx$

d) $\displaystyle\int_1^\infty \frac{x}{1+x^3}\,dx$

e) $\displaystyle\int_2^\infty \frac{x+x^2}{x^3-\cos^2(\sqrt{e^x})}\,dx$

f) $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$

g) $\displaystyle\int_1^\infty \frac{x^4}{x^5+\ln(x)}\,dx$

h) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{x}\,dx$

Løsning: Oppgave 4

a) It always holds that $0 \leq 1-\cos(x) \leq 2$ so

$$\int_e^\infty \frac{1-\cos(x)}{e^{x^2}}\,dx \leq \int_e^\infty 2e^{-x^2}\,dx.$$

On (e,∞), $e^{-x^2} \leq e^{-x}$ so we can estimate

$$\int_e^\infty 2e^{-x^2}\,dx \leq 2\int_e^\infty e^{-x}\,dx = e^{-e} < \infty.$$

b) We compare with $\frac{1}{x}$ to show divergence since

$$\begin{aligned} x-e^{-x} &< x \\ \implies \frac{1}{x-e^{-x}} &> \frac{1}{x} \\ \implies \int_{10}^\infty \frac{1}{x-e^{-x}}\,dx &> \int_{10}^\infty \frac{1}{x}\,dx \\ \implies \int_{10}^\infty \frac{1}{x-e^{-x}}\,dx &> \lim_{M\to\infty} \ln(M)-\ln(10) = \infty. \end{aligned}$$

c) Recall that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \leq 1$, hence by the mean value theorem,

$$\frac{\arctan(x) - \arctan(0)}{x - 0} = \frac{\arctan(x)}{x} \leq 1 \implies \arctan(x) \leq \frac{1}{x} \implies \frac{1}{\arctan(x)} \geq \frac{1}{x}.$$

Consequently,

$$\int_0^1 \frac{1}{\arctan(x)}\,dx \geq \int_0^1 \frac{1}{x}\,dx = \infty$$

and the integral must diverge.

d) We can estimate

$$\int_1^\infty \frac{x}{1+x^3}\,dx = \int_1^\infty \frac{1}{\frac{1}{x}+x^2}\,dx < \int_1^\infty \frac{1}{x^2}\,dx < \infty.$$

e)We split the integral into a sum of two integrals

$$\int_2^\infty \frac{x+x^2}{x^3-\cos^2(\sqrt{e^x})}\,dx = \int_2^\infty \frac{x}{x^3-\cos^2(\sqrt{e^x})}\,dx + \int_2^\infty \frac{x^2}{x^3-\cos^2(\sqrt{e^x})}\,dx.$$

Note that $\cos^2(\sqrt{e^x}) \in [0,1]$, so $\dfrac{1}{x^3-\cos^2(\sqrt{e^x})} \geq \dfrac{1}{x^3}$. Also note that $x^3 - \cos^2(\sqrt{e^x}) \geq (x-1)^3$ for $x \geq 2$. We check the two integrals individually. For the first integral we get

$$\int_2^\infty \frac{x}{x^3-\cos^2(\sqrt{e^x})}\,dx \leq \int_2^\infty \frac{x}{(x-1)^3}\,dx = \int_1^\infty \frac{x+1}{x^3}\,dx = \int_1^\infty \frac{1}{x^2}+\frac{1}{x^3}\,dx < \infty.$$

So the first integral converges. For the second integral we get

$$\int_2^\infty \frac{x^2}{x^3-\cos^2(\sqrt{e^x})}\,dx \geq \int_2^\infty \frac{x^2}{x^3}\,dx = \int_2^\infty \frac{1}{x}\,dx.$$

So the second integral, and therefore also the original integral, diverges.

f)

$$\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx < \int_1^\infty \frac{1}{x^2}\,dx < \infty$$

g) We know that $0 \leq \dfrac{\ln(x)}{x^4} \leq C$ for some constant C whenever $x \geq 1$. Therefore $x + \dfrac{\ln(x)}{x^4} \leq x + C$ for $x \geq 1$. We get in turn

$$\int_1^\infty \frac{x^4}{x^5+\ln(x)}\,dx = \int_1^\infty \frac{1}{x+\frac{\ln(x)}{x^4}}\,dx \geq \int_1^\infty \frac{1}{x+C}\,dx = \int_{1+C}^\infty \frac{1}{x}\,dx.$$

The integral diverges.

h) For $x \in [0, \frac{\pi}{4}]$ we have that $\frac{1}{\sqrt{2}} \leq \cos(x) \leq 1$. This implies

$$\int_0^{\frac{\pi}{4}} \frac{\cos(x)}{x}\,dx > \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{4}} \frac{1}{x}\,dx.$$

The integral therefore diverges since $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{x}\,dx$ diverges.