There are other equivalent definitions of the conic sections. One of these refers to special points known as *foci* (singular: *focus*). An ellipse may be defined as the set of all points in a plane the sum of whose distances d_1 and d_2 from two fixed points F_1 and F_2 (the foci) is

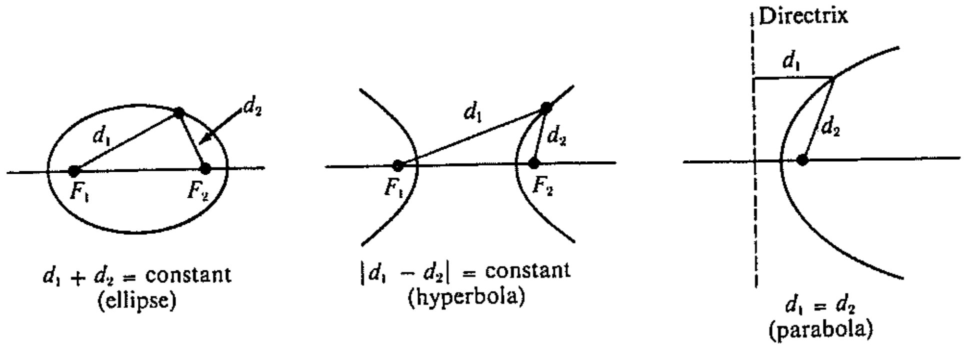
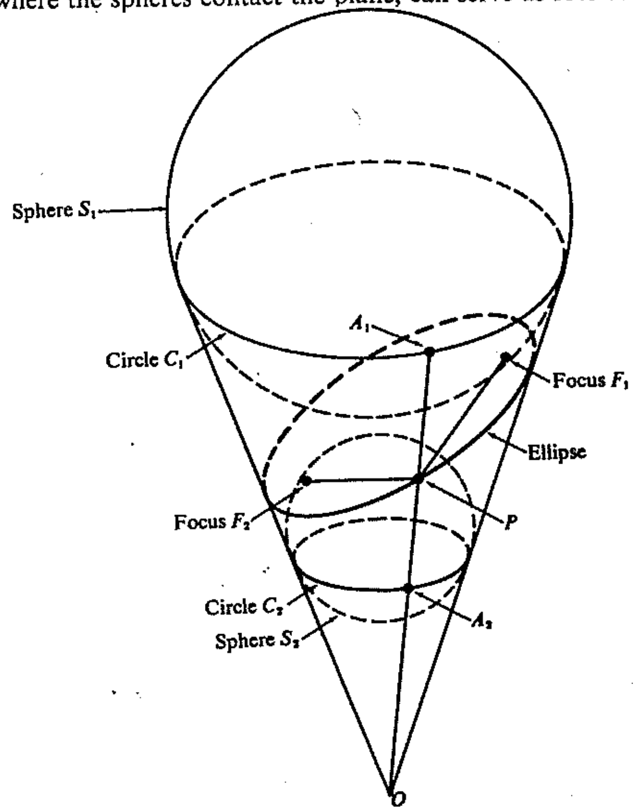


FIGURE 13.10 Focal definitions of the conic sections.

constant. (See Figure 13.10.) If the foci coincide, the ellipse reduces to a circle. A hyperbola is the set of all points for which the difference $|d_1 - d_2|$ is constant. A parabola is the set of all points in a plane for which the distance to a fixed point F (called the focus) is equal to the distance to a given line (called the directrix).

There is a very simple and elegant argument which shows that the focal property of an ellipse is a consequence of its definition as a section of a cone. This proof, which we may refer to as the "ice-cream-cone proof," was discovered in 1822 by a Belgian mathematician, G. P. Dandelin (1794–1847), and makes use of the two spheres S_1 and S_2 which are drawn so as to be tangent to the cutting plane and the cone, as illustrated in Figure 13.11. These spheres touch the cone along two parallel circles C_1 and C_2. We shall prove that the points F_1 and F_2, where the spheres contact the plane, can serve as foci of the ellipse.



Let P be an arbitrary point of the ellipse. The problem is to prove that $\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\|$ is constant, that is, independent of the choice of P. For this purpose, draw that line on the cone from the vertex O to P and let A_1 and A_2 be its intersections with the circles C_1 and C_2, respectively. Then $\overrightarrow{PF_1}$ and $\overrightarrow{PA_1}$ are two tangents to S_1 from P, and hence $\|\overrightarrow{PF_1}\| = \|\overrightarrow{PA_1}\|$. Similarly $\|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_2}\|$, and therefore we have

$$\|\overrightarrow{PF_1}\| + \|\overrightarrow{PF_2}\| = \|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| .$$

But $\|\overrightarrow{PA_1}\| + \|\overrightarrow{PA_2}\| = \|\overrightarrow{A_1A_2}\|$, which is the distance between the parallel circles C_1 and C_2 measured along the surface of the cone. This proves that F_1 and F_2 can serve as foci of the ellipse, as asserted.

Modifications of this proof work also for the hyperbola and the parabola.