

EXAMPLE 8

Solution Since $|-4|^{1/4} = \sqrt{2}$ and $\arg(-4) = \pi$, the principal 4th root of -4 is

$$w_1 = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) = 1 + i.$$

The other three 4th roots are at the vertices of a square with centre at the origin and one vertex at $1 + i$. (See Figure I.10.) Thus, the other roots are

$$w_2 = -1 + i, \quad w_3 = -1 - i, \quad w_4 = 1 - i.$$

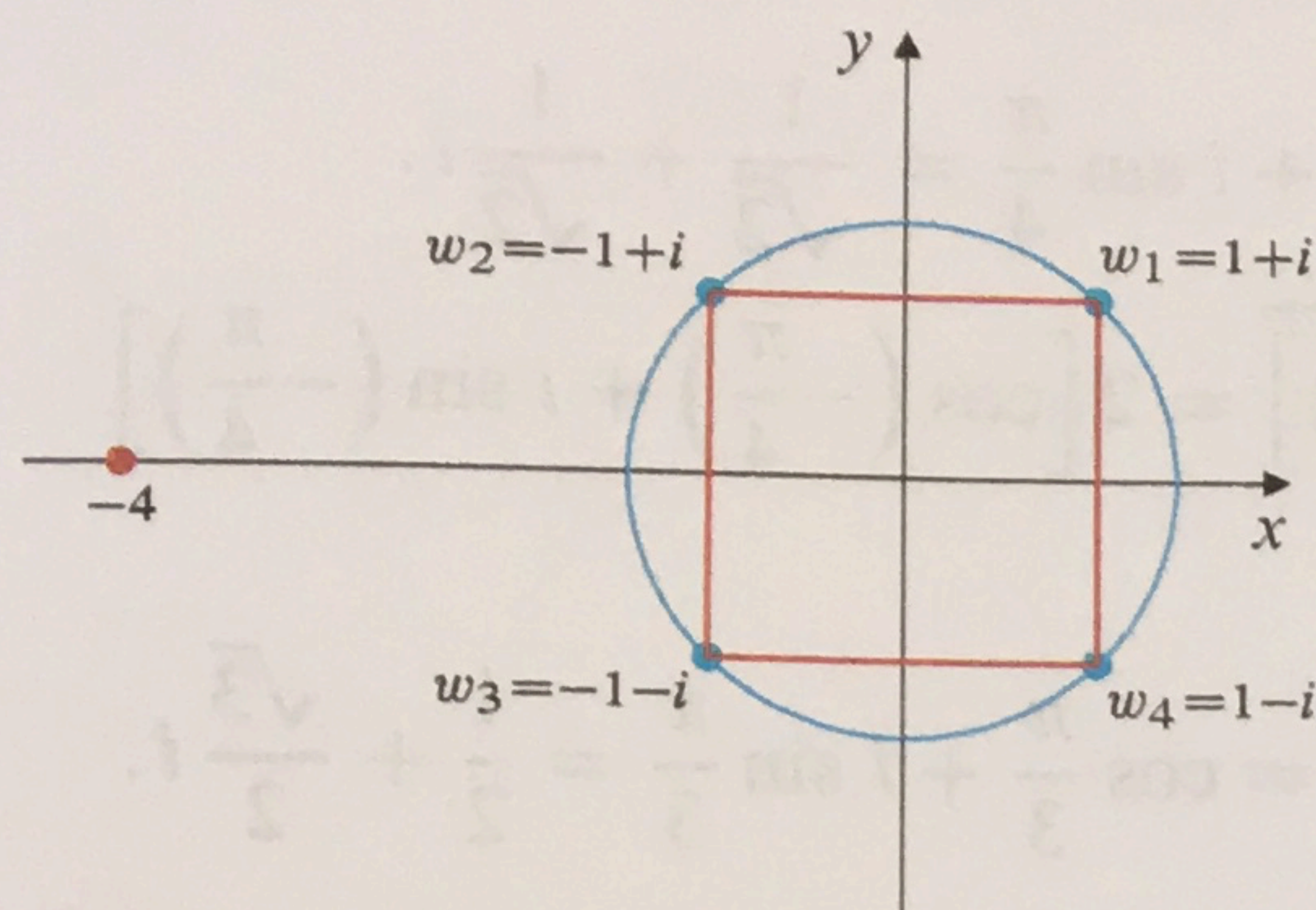


Figure I.10 The four 4th roots of -4

EXERCISES: APPENDIX I

In Exercises 1–4, find the real and imaginary parts ($\text{Re}(z)$ and $\text{Im}(z)$) of the given complex numbers z , and sketch the position of each number in the complex plane (i.e., in an Argand diagram).

- 1. $z = -5 + 2i$
- 2. $z = 4 - i$
- 3. $z = -\pi i$
- 4. $z = -6$

In Exercises 5–15, find the modulus $r = |z|$ and the principal argument $\theta = \text{Arg}(z)$ of each given complex number z , and express z in terms of r and θ .

- 5. $z = -1 + i$
- 6. $z = -2$
- 7. $z = 3i$
- 8. $z = -5i$
- 9. $z = 1 + 2i$
- 10. $z = -2 + i$
- 11. $z = -3 - 4i$
- 12. $z = 3 - 4i$
- 13. $z = \sqrt{3} - i$
- 14. $z = -\sqrt{3} - 3i$
- 15. $z = 3 \cos \frac{4\pi}{5} + 3i \sin \frac{4\pi}{5}$

- 16. If $\text{Arg}(z) = 3\pi/4$ and $\text{Arg}(w) = \pi/2$, find $\text{Arg}(zw)$.
- 17. If $\text{Arg}(z) = -5\pi/6$ and $\text{Arg}(w) = \pi/4$, find $\text{Arg}(z/w)$.

In Exercises 18–23, express in the form $z = x + yi$ the complex number z whose modulus and argument are given.

- 18. $|z| = 2, \quad \arg(z) = \pi$
- 19. $|z| = 5, \quad \arg(z) = \tan^{-1} \frac{3}{4}$
- 20. $|z| = 1, \quad \arg(z) = \frac{3\pi}{4}$
- 21. $|z| = \pi, \quad \arg(z) = \frac{\pi}{6}$
- 22. $|z| = 0, \quad \arg(z) = 1$
- 23. $|z| = \frac{1}{2}, \quad \arg(z) = -\frac{\pi}{3}$

In Exercises 24–27, find the complex conjugates of the given complex numbers.

- 24. $5 + 3i$
- 25. $-3 - 5i$
- 26. $4i$
- 27. $2 - i$

Describe geometrically (or make a sketch of) the set of points z in the complex plane satisfying the given equations or inequalities in Exercises 28–33.

- 28. $|z| = 2$
- 29. $|z| \leq 2$
- 30. $|z - 2i| \leq 3$
- 31. $|z - 3 + 4i| \leq 5$

32. $\arg z = \frac{\pi}{3}$

33. $\pi \leq \arg(z) \leq \frac{7\pi}{4}$

Simplify the expressions in Exercises 34–43.

- 34. $(2 + 5i) + (3 - i)$
- 35. $i - (3 - 2i) + (7 - 3i)$
- 36. $(4 + i)(4 - i)$
- 37. $(1 + i)(2 - 3i)$
- 38. $(a + bi)(\overline{2a - bi})$
- 39. $(2 + i)^3$
- 40. $\frac{2 - i}{2 + i}$
- 41. $\frac{1 + 3i}{2 - i}$
- 42. $\frac{1 + i}{i(2 + 3i)}$
- 43. $\frac{(1 + 2i)(2 - 3i)}{(2 - i)(3 + 2i)}$

44. Prove that $\overline{z + w} = \bar{z} + \bar{w}$.

45. Prove that $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$.

46. Express each of the complex numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in polar form (i.e., in terms of its modulus and argument). Use these expressions to calculate zw and z/w .

47. Repeat Exercise 46 for $z = -1 + i$ and $w = 3i$.

48. Use de Moivre's Theorem to find a trigonometric identity for $\cos 3\theta$ in terms of $\cos \theta$ and one for $\sin 3\theta$ in terms of $\sin \theta$.

49. Describe the solutions, if any, of the equations (a) $\bar{z} = 2/z$ and (b) $\bar{z} = -2/z$.

50. For positive real numbers a and b it is always true that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Does a similar identity hold for \sqrt{zw} , where z and w are complex numbers? *Hint:* Consider $z = w = -1$.

51. Find the three cube roots of -1 .

52. Find the three cube roots of $-8i$.

53. Find the three cube roots of $-1 + i$.

54. Find all the fourth roots of 4 .

55. Find all complex solutions of the equation $z^4 + 1 - i\sqrt{3} = 0$.

56. Find all solutions of $z^5 + a^5 = 0$, where a is a positive real number.

! 57. Show that the sum of the n n th roots of unity is zero. *Hint:* Show that these roots are all powers of the principal root.