(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

4. Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

7. The diagonals of any parallelogram bisect each other.

8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B? For what values of k is the trip not possible?

12. A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?

13. For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

14. Find the angle between a diagonal of a cube and one of the edges of the cube.

15. Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers.

16. **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

17. Find a unit vector that makes equal angles with the three coordinate axes.

18. Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

19. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20. Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

21. Let $\mathbf{a}$ be a nonzero vector, and let b be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

22. Find two unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

23. Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

24. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

26. Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are arbitrary real numbers.

27. **(The triangle inequality)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

28. (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

29. **(Orthonormal bases)** Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

30. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

31. **(Resolving a vector in perpendicular directions)** If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

32. **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}$ are position vectors of points U, V, and P, respectively, that $\mathbf{u}$ is not parallel to $\mathbf{v}$, and that P lies in the plane containing the origin, U, and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both $\mathbf{v}$ and $\mathbf{r}$ as sums of vectors parallel and perpendicular to $\mathbf{u}$ as suggested in Exercise 31.

33. Given constants r, s, and t, with $r \ne 0$ and $s \ne 0$, and given a vector $\mathbf{a}$ satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

Hanging cables

34. **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is δgx rather than δgs), show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

10.3 The Cross Product in 3-Space

There is defined, *in 3-space only*, another kind of product of two vectors called a *cross product* or *vector product*, and denoted $\mathbf{u} \times \mathbf{v}$.

DEFINITION 5

For any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, the **cross product** $\mathbf{u} \times \mathbf{v}$ is the unique vector satisfying the following three conditions:

(i) $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{v} = 0$,

(ii) $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and

(iii) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a right-handed triad.

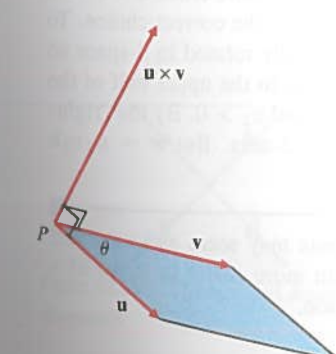

Figure 10.22 $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and has length equal to the area of the blue shaded parallelogram

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, condition (ii) says that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, the zero vector. Otherwise, through any point in $\mathbb{R}^3$ there is a unique straight line that is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Condition (i) says that $\mathbf{u} \times \mathbf{v}$ is parallel to this line. Condition (iii) determines which of the two directions along this line is the direction of $\mathbf{u} \times \mathbf{v}$; a right-handed screw advances in the direction of $\mathbf{u} \times \mathbf{v}$ if rotated in the direction from $\mathbf{u}$ toward $\mathbf{v}$. (This is equivalent to saying that the thumb, forefinger, and middle finger of the right hand can be made to point in the directions of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$, respectively.)

If $\mathbf{u}$ and $\mathbf{v}$ have their tails at the point P, then $\mathbf{u} \times \mathbf{v}$ is normal (i.e., perpendicular) to the plane through P in which $\mathbf{u}$ and $\mathbf{v}$ lie and, by condition (ii), $\mathbf{u} \times \mathbf{v}$ has length equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. (See Figure 10.22.) These properties make the cross product very useful for the description of tangent planes and normal lines to surfaces in $\mathbb{R}^3$.

The definition of cross product given above does not involve any coordinate system and therefore does not directly show the components of the cross product with respect to the standard basis. These components are provided by the following theorem.

THEOREM 2

Components of the cross product

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$