## EXERCISES 18.3

A computer is almost essential for doing most of these exercises. The calculations are easily done with a spreadsheet program in which formulas for calculating the various quantities involved can be replicated down columns to automate the iteration process.

1. Use the Euler method with step sizes (a) $h = 0.2$, (b) $h = 0.1$, and (c) $h = 0.05$ to approximate $y(2)$ given that $y' = x + y$ and $y(1) = 0$.

2. Repeat Exercise 1 using the improved Euler method.

3. Repeat Exercise 1 using the Runge–Kutta method.

4. Use the Euler method with step sizes (a) $h = 0.2$ and (b) $h = 0.1$ to approximate $y(2)$ given that $y' = xe^{-y}$ and $y(0) = 0$.

5. Repeat Exercise 4 using the improved Euler method.

6. Repeat Exercise 4 using the Runge–Kutta method.

7. Use the Euler method with (a) $h = 0.2$, (b) $h = 0.1$, and (c) $h = 0.05$ to approximate $y(1)$ given that $y' = \cos y$ and $y(0) = 0$.

8. Repeat Exercise 7 using the improved Euler method.

9. Repeat Exercise 7 using the Runge–Kutta method.

10. Use the Euler method with (a) $h = 0.2$, (b) $h = 0.1$, and (c) $h = 0.05$ to approximate $y(1)$ given that $y' = \cos(x^2)$ and $y(0) = 0$.

11. Repeat Exercise 10 using the improved Euler method.

12. Repeat Exercise 10 using the Runge–Kutta method.

Solve the integral equations in Exercises 13–14 by rephrasing them as initial-value problems.

13. $y(x) = 2 + \int_1^x \big(y(t)\big)^2\,dt$. *Hint:* Find $\dfrac{dy}{dx}$ and $y(1)$.

14. $u(x) = 1 + 3\int_2^x t^2 u(t)\,dt$. *Hint:* Find $\dfrac{du}{dx}$ and $u(2)$.

15. The methods of this section can be used to approximate definite integrals numerically. For example,
$$I = \int_a^b f(x)\,dx$$
is given by $I = y(b)$, where
$$y' = f(x), \qquad \text{and} \qquad y(a) = 0.$$
Show that one step of the Runge–Kutta method with $h = b - a$ gives the same result for $I$ as Simpson's Rule (Section 6.7) with two subintervals of length $h/2$.

16. If $\phi(0) = A \ge 0$ and $\phi'(x) \ge k\phi(x)$ on $[0, X]$, where $k > 0$ and $X > 0$ are constants, show that $\phi(x) \ge Ae^{kx}$ on $[0, X]$. *Hint:* Calculate $(d/dx)(\phi(x)/e^{kx})$.

17. Consider the three initial-value problems

| | | |
|---|---|---|
| (A) | $u' = u^2$ | $u(0) = 1$ |
| (B) | $y' = x + y^2$ | $y(0) = 1$ |
| (C) | $v' = 1 + v^2$ | $v(0) = 1$ |

(a) Show that the solution of (B) remains between the solutions of (A) and (C) on any interval $[0, X]$ where solutions of all three problems exist. *Hint:* We must have $u(x) \ge 1$, $y(x) \ge 1$, and $v(x) \ge 1$ on $[0, X]$. (Why?) Apply the result of Exercise 16 to $\phi = y - u$ and to $\phi = v - y$.