EXERCISES 3.7

In Exercises 1–12, find the general solutions for the given second-order equations.

1. $y'' + 7y' + 10y = 0$

2. $y'' - 2y' - 3y = 0$

3. $y'' + 2y' = 0$

4. $4y'' - 4y' - 3y = 0$

5. $y'' + 8y' + 16y = 0$

6. $y'' - 2y' + y = 0$

7. $y'' - 6y' + 10y = 0$

8. $9y'' + 6y' + y = 0$

9. $y'' + 2y' + 5y = 0$

10. $y'' - 4y' + 5y = 0$

11. $y'' + 2y' + 3y = 0$

12. $y'' + y' + y = 0$

In Exercises 13–15, solve the given initial-value problems.

13. $\begin{cases} 2y'' + 5y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = 0. \end{cases}$

14. $\begin{cases} y'' + 10y' + 25y = 0 \\ y(1) = 0 \\ y'(1) = 2. \end{cases}$

15. $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = 2. \end{cases}$

16. Show that if $\epsilon \neq 0$, the function $y_\epsilon(t) = \dfrac{e^{(1+\epsilon)t} - e^t}{\epsilon}$ satisfies the equation $y'' - (2+\epsilon)y' + (1+\epsilon)y = 0$. Calculate $y(t) = \lim_{\epsilon \to 0} y_\epsilon(t)$ and verify that, as expected, it is a solution of $y'' - 2y' + y = 0$.

17. If $a > 0$, $b > 0$, and $c > 0$, prove that all solutions of the differential equation $ay'' + by' + cy = 0$ satisfy $\lim_{t\to\infty} y(t) = 0$.

18. Prove that the solution given in the discussion of Case I, namely, $y = A\,e^{r_1 t} + B\,e^{r_2 t}$, is the general solution for that case as follows: First, let $y = e^{r_1 t}u$ and show that u satisfies the equation

$$u'' - (r_2 - r_1)u' = 0.$$

Then let $v = u'$, so that v must satisfy $v' = (r_2 - r_1)v$. The general solution of this equation is $v = C\,e^{(r_2 - r_1)t}$, as shown in the discussion of the equation $y' = ky$ in Section 3.4. Hence, find u and y.

Simple harmonic motion

Exercises 19–22 all refer to the differential equation of simple harmonic motion:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad (\omega \neq 0). \qquad (\dagger)$$

Together they show that $y = A\cos\omega t + B\sin\omega t$ is a *general solution* of this equation, that is, every solution is of this form for some choice of the constants A and B.

19. Show that $y = A\cos\omega t + B\sin\omega t$ is a solution of (†).

20. If $f(t)$ is any solution of (†), show that $\omega^2(f(t))^2 + (f'(t))^2$ is constant.

21. If $g(t)$ is a solution of (†) satisfying $g(0) = g'(0) = 0$, show that $g(t) = 0$ for all t.

22. Suppose that $f(t)$ is any solution of the differential equation (†). Show that $f(t) = A\cos\omega t + B\sin\omega t$, where $A = f(0)$ and $B\omega = f'(0)$.
(*Hint:* Let $g(t) = f(t) - A\cos\omega t - B\sin\omega t$.)

23. If $b^2 - 4ac < 0$, show that the substitution $y = e^{kt}u(t)$, where $k = -b/(2a)$, transforms $ay'' + by' + cy = 0$ into the equation $u'' + \omega^2 u = 0$, where $\omega^2 = (4ac - b^2)/(4a^2)$. Together with the result of Exercise 22, this confirms the recipe for Case III, in case you didn't feel comfortable with the complex number argument given in the text.

In Exercises 24–25, solve the given initial-value problems. For each problem determine the circular frequency, the frequency, the period, and the amplitude of the solution.

24. $\begin{cases} y'' + 4y = 0 \\ y(0) = 2 \\ y'(0) = -5. \end{cases}$

25. $\begin{cases} y'' + 100y = 0 \\ y(0) = 0 \\ y'(0) = 3. \end{cases}$

26. Show that $y = \alpha\cos(\omega(t-c)) + \beta\sin(\omega(t-c))$ is a solution of the differential equation $y'' + \omega^2 y = 0$, and that it satisfies $y(c) = \alpha$ and $y'(c) = \beta\omega$. Express the solution in the form $y = A\cos(\omega t) + B\sin(\omega t)$ for certain values of the constants A and B depending on α, β, c, and ω.