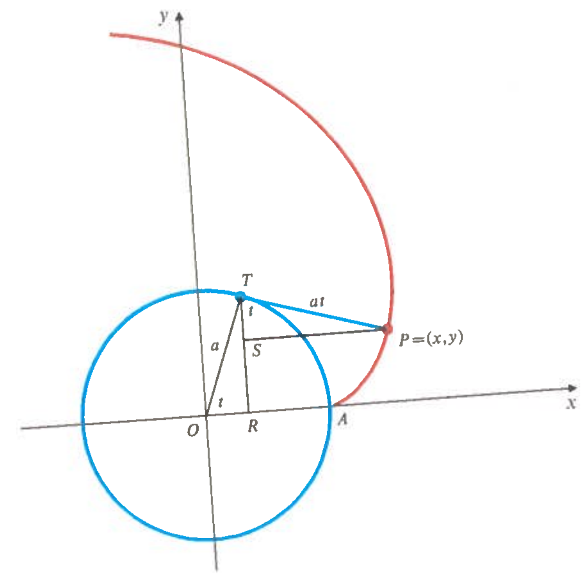

Figure 8.22 An involute of a circle

EXERCISES 8.2

In Exercises 1–10, sketch the given parametric curve, showing its direction with an arrow. Eliminate the parameter to give a Cartesian equation in x and y whose graph contains the parametric curve.

1. $x = 1 + 2t$, $y = t^2$, $(-\infty < t < \infty)$
2. $x = 2 - t$, $y = t + 1$, $(0 \le t < \infty)$
3. $x = \frac{1}{t}$, $y = t - 1$, $(0 < t < 4)$
4. $x = \frac{1}{1+t^2}$, $y = \frac{t}{1+t^2}$, $(-\infty < t < \infty)$
5. $x = 3\sin 2t$, $y = 3\cos 2t$, $\left(0 \le t \le \frac{\pi}{3}\right)$
6. $x = a\sec t$, $y = b\tan t$, $\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$
7. $x = 3\sin \pi t$, $y = 4\cos \pi t$, $(-1 \le t \le 1)$
8. $x = \cos\sin s$, $y = \sin\sin s$, $(-\infty < s < \infty)$
9. $x = \cos^3 t$, $y = \sin^3 t$, $(0 \le t \le 2\pi)$
10. $x = 1 - \sqrt{4 - t^2}$, $y = 2 + t$, $(-2 \le t \le 2)$
11. Describe the parametric curve $x = \cosh t$, $y = \sinh t$, and find its Cartesian equation.
12. Describe the parametric curve $x = 2 - 3\cosh t$, $y = -1 + 2\sinh t$.
13. Describe the curve $x = t\cos t$, $y = t\sin t$, $(0 \le t \le 4\pi)$.
14. Show that each of the following sets of parametric equations represents a different arc of the parabola with equation $2(x + y) = 1 + (x - y)^2$.
    (a) $x = \cos^4 t$, $y = \sin^4 t$
    (b) $x = \sec^4 t$, $y = \tan^4 t$
    (c) $x = \tan^4 t$, $y = \sec^4 t$
15. Find a parametrization of the parabola $y = x^2$ using as parameter the slope of the tangent line at the general point.
16. Find a parametrization of the circle $x^2 + y^2 = R^2$ using as parameter the slope m of the line joining the general point to the point $(R, 0)$. Does the parametrization fail to give any point on the circle?
17. A circle of radius a is centred at the origin O. T is a point on the circle such that OT makes angle t with the positive x-axis. The tangent to the circle at T meets the x-axis at X. The point $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through T. Find, in terms of the parameter t, parametric equations for the curve $\mathcal{C}$ traced out by P as T moves around the circle. Also, eliminate t and find an equation for $\mathcal{C}$ in x and y. Sketch $\mathcal{C}$.
18. Repeat Exercise 17 with the following modification: OT meets a second circle of radius b centred at O at the point Y. $P = (x, y)$ is at the intersection of the vertical line through X and the horizontal line through Y.
19. **(The folium of Descartes)** Eliminate the parameter from the parametric equations

$$x = \frac{3t}{1+t^3}, \qquad y = \frac{3t^2}{1+t^3}, \qquad (t \ne -1),$$

and hence find an ordinary equation in x and y for this curve. The parameter t can be interpreted as the slope of the line joining the general point (x, y) to the origin. Sketch the curve and show that the line $x + y = -1$ is an asymptote.

20. **(A prolate cycloid)** A railroad wheel has a flange extending below the level of the track on which the wheel rolls. If the radius of the wheel is a and that of the flange is $b > a$, find parametric equations of the path of a point P at the circumference of the flange as the wheel rolls along the track. (Note that for a portion of each revolution of the wheel, P is moving backward.) Try to sketch the graph of this prolate cycloid.
21. **(Hypocycloids)** If a circle of radius b rolls, without slipping, around the inside of a fixed circle of radius $a > b$, a point on the circumference of the rolling circle traces a curve called a hypocycloid. If the fixed circle is centred at the origin and the point tracing the curve starts at $(a, 0)$, show that the hypocycloid has parametric equations

$$x = (a - b)\cos t + b\cos\left(\frac{a-b}{b}t\right),$$
$$y = (a - b)\sin t - b\sin\left(\frac{a-b}{b}t\right),$$

where t is the angle between the positive x-axis and the line from the origin to the point at which the rolling circle touches the fixed circle.

If $a = 2$ and $b = 1$, show that the hypocycloid becomes a straight line segment.

If $a = 4$ and $b = 1$, show that the parametric equations of the hypocycloid simplify to $x = 4\cos^3 t$, $y = 4\sin^3 t$. This curve is called a hypocycloid of four cusps or an **astroid**. (See Figure 8.23.) It has Cartesian equation $x^{2/3} + y^{2/3} = 4^{2/3}$.

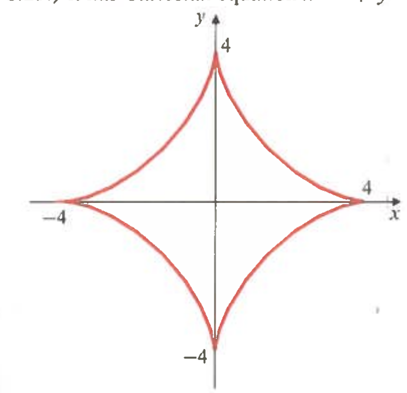

Figure 8.23 The astroid $x^{2/3} + y^{2/3} = 4^{2/3}$

Hypocycloids resemble the curves produced by a popular children's toy called Spirograph, but Spirograph curves result from following a point inside the disc of the rolling circle rather than on its circumference, and they therefore do not have sharp cusps.

22. **(The witch of Agnesi)**
    (a) Show that the curve traced out by the point P constructed from a circle as shown in Figure 8.24 has parametric equations $x = \tan t$, $y = \cos^2 t$ in terms of the angle t shown. (*Hint:* You will need to make extensive use of similar triangles.)
    (b) Use a trigonometric identity to eliminate t from the parametric equations, and hence find an ordinary Cartesian equation for the curve.

This curve is named for the Italian mathematician Maria Agnesi (1718–1799), one of the foremost women scholars of the eighteenth century and author of an important calculus text. The term *witch* is due to a mistranslation of the Italian word *versiera* ("turning curve"), which she used to describe the curve. The word is similar to *avversiera* ("wife of the devil" or "witch").

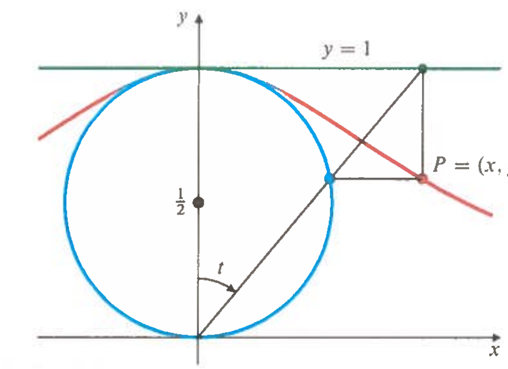

Figure 8.24 The witch of Agnesi

In Exercises 23–26, obtain a graph of the curve $x = \sin(mt)$, $y = \sin(nt)$ for the given values of m and n. Such curves are called **Lissajous figures**. They arise in the analysis of electrical signals using an oscilloscope. A signal of fixed but unknown frequency is applied to the vertical input, and a control signal is applied to the horizontal input. The horizontal frequency is varied until a stable Lissajous figure is observed. The (known) frequency of the control signal and the shape of the figure then determine the unknown frequency.

23. $m = 1$, $n = 2$
24. $m = 1$, $n = 3$
25. $m = 2$, $n = 3$
26. $m = 2$, $n = 5$
27. **(Epicycloids)** Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t - \frac{1}{n}\cos(nt)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin(nt)$$

for various integer and fractional values of $n \ge 3$. Can you formulate any principles governing the behaviour of such curves?

28. **(More hypocycloids)** Use a graphing calculator or computer graphing program to investigate the behaviour of curves with equations of the form

$$x = \left(1 + \frac{1}{n}\right)\cos t + \frac{1}{n}\cos((n-1)t)$$
$$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin((n-1)t)$$

for various integer and fractional values of $n \ge 3$. Can you formulate any principles governing the behaviour of these curves?

8.3 Smooth Parametric Curves and Their Slopes

We say that a plane curve is *smooth* if it has a tangent line at each point P and this tangent turns in a continuous way as P moves along the curve. (That is, the angle