Sketching Parametric Curves

As in the case of graphs of functions, derivatives provide useful information about the shape of a parametric curve. At points where $dy/dt = 0$ but $dx/dt \neq 0$, the tangent is horizontal; at points where $dx/dt = 0$ but $dy/dt \neq 0$, the tangent is vertical. For points where $dx/dt = dy/dt = 0$, anything can happen; it is wise to calculate left- and right-hand limits of the slope dy/dx as the parameter t approaches one of these points. Concavity can be determined using the formula obtained above. We illustrate these ideas by reconsidering a parametric curve encountered in the previous section.

EXAMPLE 4 Use slope and concavity information to sketch the graph of the parametric curve

$$x = f(t) = t^3 - 3t, \qquad y = g(t) = t^2, \qquad (-2 \le t \le 2)$$

previously encountered in Example 5 of Section 8.2.

Solution We have

$$f'(t) = 3(t^2 - 1) = 3(t-1)(t+1), \qquad g'(t) = 2t.$$

The curve has a horizontal tangent at $t = 0$, that is, at $(0, 0)$, and vertical tangents at $t = \pm 1$, that is, at $(2, 1)$ and $(-2, 1)$. Directional information for the curve between these points is summarized in the following chart.

t	-2		-1		0		1		2
$f'(t)$		+	0	−	−	−	0	+	
$g'(t)$		−	−	−	0	+	+	+	
x		→	·	←	←	←	·	→	
y		↓	↓	↓	·	↑	↑	↑	
curve		↘	↓	↙	←	↖	↑	↗	

For concavity we calculate the second derivative d^2y/dx^2 by the formula obtained above. Since $f''(t) = 6t$ and $g''(t) = 2$, we have

$$\frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3}$$
$$= \frac{3(t^2-1)(2) - 2t(6t)}{[3(t^2-1)]^3} = -\frac{2}{9}\frac{t^2+1}{(t^2-1)^3},$$

which is never zero but which fails to be defined at $t = \pm 1$. Evidently the curve is concave upward for $-1 < t < 1$ and concave downward elsewhere. The curve is sketched in Figure 8.26.

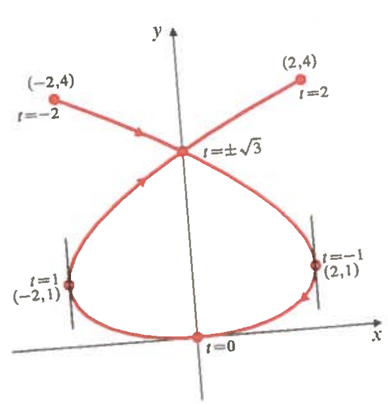

Figure 8.26 The curve $x = t^3 - 3t$, $y = t^2$, $(-2 \le t \le 2)$

EXERCISES 8.3

In Exercises 1–8, find the coordinates of the points at which the given parametric curve has (a) a horizontal tangent and (b) a vertical tangent.

1. $x = t^2 + 1, \ y = 2t - 4$ **2.** $x = t^2 - 2t, \ y = t^2 + 2t$

3. $x = t^2 - 2t, \ y = t^3 - 12t$

4. $x = t^3 - 3t, \ y = 2t^3 + 3t^2$

5. $x = te^{-t^2/2}, \ y = e^{-t^2}$

6. $x = \sin t, \ y = \sin t - t\cos t$

7. $x = \sin 2t, \ y = \sin t$ **8.** $x = \dfrac{3t}{1+t^3}, \ y = \dfrac{3t^2}{1+t^3}$

Find the slopes of the curves in Exercises 9–12 at the points indicated.

9. $x = t^3 + t, \ y = 1 - t^3$, at $t = 1$

10. $x = t^4 - t^2, \ y = t^3 + 2t$, at $t = -1$

11. $x = \cos 2t, \ y = \sin t$, at $t = \pi/6$

12. $x = e^{2t}, \ y = te^{2t}$, at $t = -2$

Find parametric equations of the tangents to the curves in Exercises 13–14 at the indicated points.

13. $x = t^3 - 2t, \ y = t + t^3$, at $t = 1$

14. $x = t - \cos t, \ y = 1 - \sin t$, at $t = \pi/4$

15. Show that the curve $x = t^3 - t$, $y = t^2$ has two different tangent lines at the point $(0, 1)$ and find their slopes.

16. Find the slopes of two lines that are tangent to $x = \sin t$, $y = \sin 2t$ at the origin.

Where, if anywhere, do the curves in Exercises 17–20 fail to be smooth?

17. $x = t^3, \ y = t^2$

18. $x = (t-1)^4, \ y = (t-1)^3$

19. $x = t\sin t, \ y = t^3$ **20.** $x = t^3, \ y = t - \sin t$

In Exercises 21–25, sketch the graphs of the given parametric curves, making use of information from the first two derivatives. Unless otherwise stated, the parameter interval for each curve is the whole real line.

21. $x = t^2 - 2t, \ y = t^2 - 4t$ **22.** $x = t^3, \ y = 3t^2 - 1$

23. $x = t^3 - 3t, \ y = \dfrac{2}{1+t^2}$

24. $x = t^3 - 3t - 2, \ y = t^2 - t - 2$

25. $x = \cos t + t\sin t, \ y = \sin t - t\cos t, \ (t \ge 0)$. (See Example 9 of Section 8.2.)

8.4 Arc Lengths and Areas for Parametric Curves

In this section we look at the problems of finding lengths of curves defined parametrically, areas of surfaces of revolution obtained by rotating parametric curves, and areas of plane regions bounded by parametric curves.

Arc Lengths and Surface Areas

Let $\mathscr{C}$ be a smooth parametric curve with equations

$$x = f(t), \qquad y = g(t), \qquad (a \le t \le b).$$

(We assume that $f'(t)$ and $g'(t)$ are continuous on the interval $[a, b]$ and are never both zero.) From the differential triangle with legs dx and dy and hypotenuse ds (see Figure 8.27), we obtain $(ds)^2 = (dx)^2 + (dy)^2$, so we have

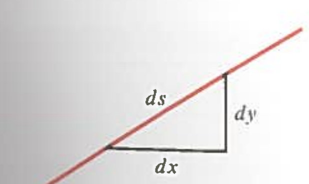

Figure 8.27 A differential triangle

The arc length element for a parametric curve

$$ds = \frac{ds}{dt}\,dt = \sqrt{\left(\frac{ds}{dt}\right)^2}\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The length of the curve $\mathscr{C}$ is given by

$$s = \int_{t=a}^{t=b} ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

EXAMPLE 1 Find the length of the parametric curve

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad (0 \le t \le 2).$$