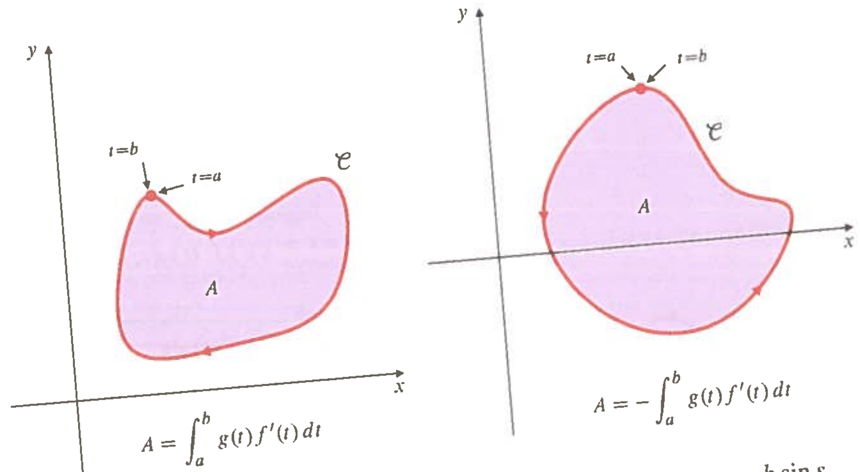

Figure 8.31 Areas bounded by closed parametric curves

EXAMPLE 3 Find the area bounded by the ellipse $x = a\cos s$, $y = b\sin s$, $(0 \le s \le 2\pi)$.

Solution This ellipse is traversed counterclockwise. (See Example 4 in Section 8.2.) The area enclosed is

$$\begin{aligned} A &= -\int_0^{2\pi} b\sin s(-a\sin s)\,ds \\ &= \frac{ab}{2}\int_0^{2\pi}(1-\cos 2s)\,ds \\ &= \frac{ab}{2}s\Big|_0^{2\pi} - \frac{ab}{4}\sin 2s\Big|_0^{2\pi} = \pi ab \text{ square units.} \end{aligned}$$

EXAMPLE 4 Find the area above the x-axis and under one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$.

Solution Part of the cycloid is shown in Figure 8.21 in Section 8.2. One arch corresponds to the parameter interval $0 \le t \le 2\pi$. Since $y = a(1-\cos t) \ge 0$ and $dx/dt = a(1-\cos t) \ge 0$, the area under one arch is

$$\begin{aligned} A &= \int_0^{2\pi} a^2(1-\cos t)^2\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt \\ &= a^2\left(t - 2\sin t + \frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_0^{2\pi} = 3\pi a^2 \text{ square units.} \end{aligned}$$

Similar arguments to those used above show that if f is continuous and g is differentiable, then we can also interpret

$$\int_a^b f(t)g'(t)\,dt = \int_{t=a}^{t=b} x\,dy = A_1 - A_2,$$

where A_1 is the area of the region lying *horizontally* between the parametric curve $x = f(t)$, $y = g(t)$, $(a \le t \le b)$ and that part of the y-axis consisting of points $y = g(t)$ such that $f(t)g'(t) \ge 0$, and A_2 is the area of a similar region corresponding to $f(t)g'(t) < 0$. For example, the region shaded in Figure 8.32 has area $\int_a^b f(t)g'(t)\,dt$. Green's Theorem in Section 16.3 provides a more coherent approach to finding such areas.

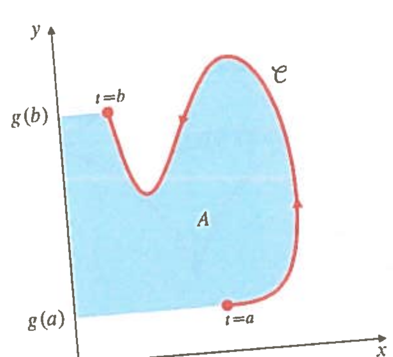

Figure 8.32 The shaded area is $A = \int_a^b f(t)g'(t)\,dt$

EXERCISES 8.4

Find the lengths of the curves in Exercises 1–8.

1. $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$
2. $x = 1 + t^3$, $y = 1 - t^2$, $(-1 \le t \le 2)$
3. $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$
4. $x = \ln(1+t^2)$, $y = 2\tan^{-1} t$, $(0 \le t \le 1)$
5. $x = t^2\sin t$, $y = t^2\cos t$, $(0 \le t \le 2\pi)$
6. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $(0 \le t \le 2\pi)$
7. $x = t + \sin t$, $y = \cos t$, $(0 \le t \le \pi)$
8. $x = \sin^2 t$, $y = 2\cos t$, $(0 \le t \le \pi/2)$
9. Find the length of one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (One arch corresponds to $0 \le t \le 2\pi$.)
10. Find the area of the surfaces obtained by rotating one arch of the cycloid in Exercise 9 about (a) the x-axis, (b) the y-axis.
11. Find the area of the surface generated by rotating the curve $x = e^t\cos t$, $y = e^t\sin t$, $(0 \le t \le \pi/2)$ about the x-axis.
12. Find the area of the surface generated by rotating the curve of Exercise 11 about the y-axis.
13. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the y-axis.
14. Find the area of the surface generated by rotating the curve $x = 3t^2$, $y = 2t^3$, $(0 \le t \le 1)$ about the x-axis.

In Exercises 15–20, sketch and find the area of the region R described in terms of the given parametric curves.

15. R is the closed loop bounded by $x = t^3 - 4t$, $y = t^2$, $(-2 \le t \le 2)$.
16. R is bounded by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$.
17. R is bounded by the coordinate axes and the parabolic arc $x = \sin^4 t$, $y = \cos^4 t$.
18. R is bounded by $x = \cos s\sin s$, $y = \sin^2 s$, $(0 \le s \le \pi/2)$, and the y-axis.
19. R is bounded by the oval $x = (2 + \sin t)\cos t$, $y = (2 + \sin t)\sin t$.
20. ! R is bounded by the x-axis, the hyperbola $x = \sec t$, $y = \tan t$, and the ray joining the origin to the point $(\sec t_0, \tan t_0)$.
21. Show that the region bounded by the x-axis and the hyperbola $x = \cosh t$, $y = \sinh t$ (where $t > 0$), and the ray from the origin to the point $(\cosh t_0, \sinh t_0)$ has area $t_0/2$ square units. This proves a claim made at the beginning of Section 3.6.
22. Find the volume of the solid obtained by rotating about the x-axis the region bounded by that axis and one arch of the cycloid $x = at - a\sin t$, $y = a - a\cos t$. (See Example 8 in Section 8.2.)
23. Find the volume generated by rotating about the x-axis the region lying under the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ and above the x-axis.

8.5 Polar Coordinates and Polar Curves

The **polar coordinate system** is an alternative to the rectangular (Cartesian) coordinate system for describing the location of points in a plane. Sometimes it is more important to know how far, and in what direction, a point is from the origin than it is to know its Cartesian coordinates. In the polar coordinate system there is an origin (or **pole**), O, and a **polar axis**, a ray (i.e., a half-line) extending from O horizontally to the right. The position of any point P in the plane is then determined by its polar coordinates $[r, \theta]$, where

(i) r is the distance from O to P, and

(ii) θ is the angle that the ray OP makes with the polar axis (counterclockwise angles being considered positive).

We will use square brackets for polar coordinates of a point to distinguish them from rectangular (Cartesian) coordinates. Figure 8.33 shows some points with their polar coordinates. The rectangular coordinate axes x and y are usually shown on a polar graph. The polar axis coincides with the positive x-axis.

Unlike rectangular coordinates, the polar coordinates of a point are not unique. The polar coordinates $[r, \theta_1]$ and $[r, \theta_2]$ represent the same point provided θ_1 and θ_2 differ by an integer multiple of 2π:

$$\theta_2 = \theta_1 + 2n\pi, \qquad \text{where } n = 0, \pm 1, \pm 2, \ldots.$$