EXERCISES 9.3

In Exercises 1–26, determine whether the given series converges or diverges by using any appropriate test. The p-series can be used for comparison, as can geometric series. Be alert for series whose terms do not approach 0.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\sum_{n=1}^{\infty} \frac{n}{n^4-2}$

3. $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$

5. $\sum_{n=1}^{\infty} \left|\sin \frac{1}{n^2}\right|$

6. $\sum_{n=8}^{\infty} \frac{1}{\pi^n+5}$

7. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^3}$

8. $\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$

9. $\sum_{n=1}^{\infty} \frac{1}{\pi^n - n^\pi}$

10. $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$

11. $\sum_{n=1}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

12. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

13. $\sum_{n=3}^{\infty} \frac{1}{n\ln n\sqrt{\ln\ln n}}$

14. $\sum_{n=2}^{\infty} \frac{1}{n\ln n(\ln\ln n)^2}$

15. $\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}$

16. $\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}}$

17. $\sum_{n=1}^{\infty} \frac{1}{2^n(n+1)}$

18. $\sum_{n=1}^{\infty} \frac{n^4}{n!}$

19. $\sum_{n=1}^{\infty} \frac{n!}{n^2e^n}$

20. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$

21. $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n\ln n}$

22. $\sum_{n=0}^{\infty} \frac{n^{100}2^n}{\sqrt{n!}}$

23. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^3}$

24. $\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$

25. $\sum_{n=4}^{\infty} \frac{2^n}{3^n-n^3}$

26. $\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$

In Exercises 27–30, use s_n and integral bounds to find the smallest interval that you can be sure contains the sum s of the series. If the midpoint s_n^* of this interval is used to approximate s, how large should n be chosen to ensure that the error is less than 0.001?

27. $\sum_{k=1}^{\infty} \frac{1}{k^4}$

28. $\sum_{k=1}^{\infty} \frac{1}{k^3}$

29. $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

30. $\sum_{k=1}^{\infty} \frac{1}{k^2+4}$

For each positive series in Exercises 31–34, find the best upper bound you can for the error $s - s_n$ encountered if the partial sum s_n is used to approximate the sum s of the series. How many terms of each series do you need to be sure that the approximation has error less than 0.001?

31. $\sum_{k=1}^{\infty} \frac{1}{2^k k!}$

32. $\sum_{n=1}^{\infty} \frac{1}{(2n-1)!}$

33. $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!}$

34. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

35. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges. Show that the sum s of the series is less than $\pi/2$.

! 36. Show that $\sum_{n=3}^{\infty}(1/(n\ln n(\ln\ln n)^p))$ converges if and only if $p > 1$. Generalize this result to series of the form

$$\sum_{n=N}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)\cdots(\ln_j n)(\ln_{j+1} n)^p},$$

where $\ln_j n = \underbrace{\ln\ln\ln\ln\cdots\ln}_{j\ \ln's} n$.

! 37. Prove the root test. *Hint:* Mimic the proof of the ratio test.

38. Use the root test to show that $\sum_{n=1}^{\infty} \frac{2^{n+1}}{n^n}$ converges.

! 39. Use the root test to test the following series for convergence:

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}.$$

40. Repeat Exercise 38, but use the ratio test instead of the root test.

! 41. Try to use the ratio test to determine whether $\sum_{n=1}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!}$ converges. What happens? Now observe that

$$\frac{2^{2n}(n!)^2}{(2n)!} = \frac{[2n(2n-2)(2n-4)\cdots 6\times 4\times 2]^2}{2n(2n-1)(2n-2)\cdots 4\times 3\times 2\times 1} = \frac{2n}{2n-1}\times\frac{2n-2}{2n-3}\times\cdots\times\frac{4}{3}\times\frac{2}{1}.$$

Does the given series converge? Why or why not?

! 42. Determine whether the series $\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}$ converges. *Hint:* Proceed as in Exercise 41. Show that $a_n \geq 1/(2n)$.

! 43. (a) Show that if $k > 0$ and n is a positive integer, then $n < \frac{1}{k}(1+k)^n$.

(b) Use the estimate in (a) with $0 < k < 1$ to obtain an upper bound for the sum of the series $\sum_{n=0}^{\infty} n/2^n$. For what value of k is this bound lowest?

(c) If we use the sum s_n of the first n terms to approximate the sum s of the series in (b), obtain an upper bound for the error $s - s_n$ using the inequality from (a). For given n, find k to minimize this upper bound.

? 44. **(Improving the convergence of a series)** We know that $\sum_{n=1}^{\infty} 1/\big(n(n+1)\big) = 1$. (See Example 3 of Section 9.2.) Since $\frac{1}{n^2} = \frac{1}{n(n+1)} + c_n$, where $c_n = \frac{1}{n^2(n+1)}$, we have $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} c_n$.