If a series converges absolutely, the subseries consisting of positive terms and the subseries consisting of negative terms must each converge to a finite sum. If a series converges conditionally, the positive and negative subseries will both diverge, to $\infty$ and $-\infty$, respectively.

Using these facts we can answer a question raised at the beginning of Section 9.2. If we rearrange the terms of a convergent series so that they are added in a different order, must the rearranged series converge, and if it does will it converge to the same sum as the original series? The answer depends on whether the original series was absolutely convergent or merely conditionally convergent.

**THEOREM 16**

**Convergence of rearrangements of a series**

(a) If the terms of an absolutely convergent series are rearranged so that addition occurs in a different order, the rearranged series still converges to the same sum as the original series.

(b) If a series is conditionally convergent, and $L$ is any real number, then the terms of the series can be rearranged so as to make the series converge (conditionally) to the sum $L$. It can also be rearranged so as to diverge to $\infty$ or to $-\infty$, or just to diverge.

Part (b) shows that conditional convergence is a rather suspect kind of convergence, being dependent on the order in which the terms are added. We will not present a formal proof of the theorem but will give an example suggesting what is involved. (See also Exercise 30 below.)

**EXAMPLE 7** In Section 9.5 we will show that the alternating harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots$$

converges (conditionally) to the sum $\ln 2$. Describe how to rearrange its terms so that it converges to 8 instead.

***Solution*** Start adding terms of the positive subseries

$$1+\frac{1}{3}+\frac{1}{5}+\cdots,$$

and keep going until the partial sum exceeds 8. (It will, eventually, because the positive subseries diverges to infinity.) Then add the first term $-1/2$ of the negative subseries

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots.$$

This will reduce the partial sum below 8 again. Now resume adding terms of the positive subseries until the partial sum climbs above 8 once more. Then add the second term of the negative subseries and the partial sum will drop below 8. Keep repeating this procedure, alternately adding terms of the positive subseries to force the sum above 8 and then terms of the negative subseries to force it below 8. Since both subseries have infinitely many terms and diverge to $\infty$ and $-\infty$, respectively, eventually every term of the original series will be included, and the partial sums of the new series will oscillate back and forth around 8, converging to that number. Of course, any number other than 8 could also be used in place of 8.



## EXERCISES 9.4

Determine whether the series in Exercises 1–12 converge absolutely, converge conditionally, or diverge.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2+\ln n}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{(n+1)\ln(n+1)}$

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2^n}$

5. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n^2-1)}{n^2+1}$

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi^n}$

8. $\displaystyle\sum_{n=0}^{\infty}\frac{-n}{n^2+1}$

9. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{20n^2-n-1}{n^3+n^2+33}$

10. $\displaystyle\sum_{n=1}^{\infty}\frac{100\cos(n\pi)}{2n+3}$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{(-100)^n}$

12. $\displaystyle\sum_{n=10}^{\infty}\frac{\sin(n+1/2)\pi}{\ln\ln n}$

For the series in Exercises 13–16, find the smallest integer $n$ that ensures that the partial sum $s_n$ approximates the sum $s$ of the series with error less than 0.001 in absolute value.

13. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$

14. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{2^n}$

16. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$

Determine the values of $x$ for which the series in Exercises 17–24 converge absolutely, converge conditionally, or diverge.

17. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n+1}}$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2 2^{2n}}$

19. $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^n}{2n+3}$

20. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{3x+2}{-5}\right)^n$

21. $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{2^n\ln n}$

22. $\displaystyle\sum_{n=1}^{\infty}\frac{(4x+1)^n}{n^3}$

23. $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+3)^n}{n^{1/3}4^n}$

24. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left(1+\frac{1}{x}\right)^n$

25. Does the alternating series test apply directly to the series $\sum_{n=1}^{\infty}(1/n)\sin(n\pi/2)$? Determine whether the series converges.

26. Show that the series $\sum_{n=1}^{\infty}a_n$ converges absolutely if $a_n=10/n^2$ for even $n$ and $a_n=-1/10n^3$ for odd $n$.

27. Which of the following statements are TRUE and which are FALSE? Justify your assertion of truth, or give a counter-example to show falsehood.
   (a) If $\sum_{n=1}^{\infty}a_n$ converges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.
   (b) If $\sum_{n=1}^{\infty}a_n$ converges and $\sum_{n=1}^{\infty}(-1)^n a_n$ converges, then $\sum_{n=1}^{\infty}a_n$ converges absolutely.
   (c) If $\sum_{n=1}^{\infty}a_n$ converges absolutely, then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges absolutely.

28. (a) Use a Riemann sum argument to show that

$$\ln n!\ge\int_1^n\ln t\,dt=n\ln n-n+1.$$

   (b) For what values of $x$ does the series $\sum_{n=1}^{\infty}\frac{n!x^n}{n^n}$ converge absolutely? converge conditionally? diverge? (*Hint:* First use the ratio test. To test the cases where $\rho=1$, you may find the inequality in part (a) useful.)

29. For what values of $x$ does the series $\sum_{n=1}^{\infty}\frac{(2n)!x^n}{2^{2n}(n!)^2}$ converge absolutely? converge conditionally? diverge? *Hint:* See Exercise 42 of Section 9.3.

30. Devise procedures for rearranging the terms of the alternating harmonic series so that the rearranged series (a) diverges to $\infty$, (b) converges to $-2$.

## 9.5 Power Series

This section is concerned with a special kind of infinite series called a *power series*, which may be thought of as a polynomial of infinite degree.