EXERCISES 9.8

Find Maclaurin series representations for the functions in Exercises 1–8. Use the binomial series to calculate the answers.

1. $\sqrt{1+x}$
2. $x\sqrt{1-x}$
3. $\sqrt{4+x}$
4. $\dfrac{1}{\sqrt{4+x^2}}$
5. $(1-x)^{-2}$
6. $(1+x)^{-3}$
7. $\cos^{-1} x$
8. $\sinh^{-1} x$

9. **(Binomial coefficients)** Show that the binomial coefficients $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ satisfy

(i) $\binom{n}{0} = \binom{n}{n} = 1$ for every n, and

(ii) if $0 \le k \le n$, then $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.

It follows that, for fixed $n \ge 1$, the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

are the elements of the nth row of **Pascal's triangle** below, where each element with value greater than 1 is the sum of the two diagonally above it.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \cdot & & & & & & & & & & \cdot \\ & & & & & \cdot & & & & & \end{array}$$

10. **(An inductive proof of the Binomial Theorem)** Use mathematical induction and the results of Exercise 9 to prove the Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$
$$= a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \cdots + b^n.$$

11. **(The Leibniz Rule)** Use mathematical induction, the Product Rule, and Exercise 9 to verify the Leibniz Rule for the nth derivative of a product of two functions:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$
$$= f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2} f^{(n-2)}g''$$
$$+ \binom{n}{3} f^{(n-3)}g^{(3)} + \cdots + fg^{(n)}.$$

12. **(Proof of the Multinomial Theorem)** Use the Binomial Theorem and induction on n to prove Theorem 24. *Hint:* Assume the theorem holds for specific n and all k. Apply the Binomial Theorem to

$$(x_1 + \cdots + x_n + x_{n+1})^k = \big((x_1 + \cdots + x_n) + x_{n+1}\big)^k.$$

13. **(A Multifunction Leibniz Rule)** Use the technique of Exercise 12 to generalize the Leibniz Rule of Exercise 11 to calculate the kth derivative of a product of n functions $f_1 f_2 \cdots f_n$.