## EXERCISES 9.6

Find Maclaurin series representations for the functions in Exercises 1–14. For what values of $x$ is each representation valid?

1. $e^{3x+1}$
2. $\cos(2x^3)$
3. $\sin(x - \pi/4)$
4. $\cos(2x - \pi)$
5. $x^2 \sin(x/3)$
6. $\cos^2(x/2)$
7. $\sin x \cos x$
8. $\tan^{-1}(5x^2)$
9. $\dfrac{1+x^3}{1+x^2}$
10. $\ln(2 + x^2)$
11. $\ln \dfrac{1+x}{1-x}$
12. $(e^{2x^2} - 1)/x^2$
13. $\cosh x - \cos x$
14. $\sinh x - \sin x$

Find the required Taylor series representations of the functions in Exercises 15–26. Where is each series representation valid?

15. $f(x) = e^{-2x}$ about $-1$
16. $f(x) = \sin x$ about $\pi/2$
17. $f(x) = \cos x$ in powers of $x - \pi$
18. $f(x) = \ln x$ in powers of $x - 3$
19. $f(x) = \ln(2 + x)$ in powers of $x - 2$
20. $f(x) = e^{2x+3}$ in powers of $x + 1$
21. $f(x) = \sin x - \cos x$ about $\dfrac{\pi}{4}$
22. $f(x) = \cos^2 x$ about $\dfrac{\pi}{8}$
23. $f(x) = 1/x^2$ in powers of $x + 2$
24. $f(x) = \dfrac{x}{1+x}$ in powers of $x - 1$
25. $f(x) = x \ln x$ in powers of $x - 1$
26. $f(x) = xe^x$ in powers of $x + 2$

Find the first three nonzero terms in the Maclaurin series for the functions in Exercises 27–30.

27. $\sec x$
28. $\sec x \tan x$
29. $\tan^{-1}(e^x - 1)$
30. $e^{\tan^{-1} x} - 1$

31. Use the fact that $(\sqrt{1+x})^2 = 1 + x$ to find the first three nonzero terms of the Maclaurin series for $\sqrt{1+x}$.

32. Does $\csc x$ have a Maclaurin series? Why? Find the first three nonzero terms of the Taylor series for $\csc x$ about the point $x = \pi/2$.

Find the sums of the series in Exercises 33–36.

33. $1 + x^2 + \dfrac{x^4}{2!} + \dfrac{x^6}{3!} + \dfrac{x^8}{4!} + \cdots$

34. $x^3 - \dfrac{x^9}{3! \times 4} + \dfrac{x^{15}}{5! \times 16} - \dfrac{x^{21}}{7! \times 64} + \dfrac{x^{27}}{9! \times 256} - \cdots$

35. $1 + \dfrac{x^2}{3!} + \dfrac{x^4}{5!} + \dfrac{x^6}{7!} + \dfrac{x^8}{9!} + \cdots$

36. $1 + \dfrac{1}{2 \times 2!} + \dfrac{1}{4 \times 3!} + \dfrac{1}{8 \times 4!} + \cdots$

37. Let $P(x) = 1 + x + x^2$. Find (a) the Maclaurin series for $P(x)$ and (b) the Taylor series for $P(x)$ about 1.

38. Verify by direct calculation that $f(x) = 1/x$ is analytic at $a$ for every $a \neq 0$.

39. Verify by direct calculation that $\ln x$ is analytic at $a$ for every $a > 0$.

40. Review Exercise 41 of Section 4.5. It shows that the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has derivatives of all orders at every point of the real line, and $f^{(k)}(0) = 0$ for every positive integer $k$. What is the Maclaurin series for $f(x)$? What is the interval of convergence of this Maclaurin series? On what interval does the series converge to $f(x)$? Is $f$ analytic at 0?

41. By direct multiplication of the Maclaurin series for $e^x$ and $e^y$ show that $e^x e^y = e^{x+y}$.

**42. (Taylor's Formula with integral remainder)** Verify that if $f^{(n+1)}$ exists on an interval containing $c$ and $x$, and if $P_n(x)$ is the $n$th-order Taylor polynomial for $f$ about $c$, then $f(x) = P_n(x) + E_n(x)$, where

$$E_n(x) = \frac{1}{n!} \int_c^x (x - t)^n f^{(n+1)}(t)\, dt.$$

Proceed as follows:

(a) First observe that the case $n = 0$ is just the Fundamental Theorem of Calculus:

$$f(x) = f(c) + \int_c^x f'(t)\, dt.$$

Now integrate by parts in this formula, taking $U = f'(t)$ and $dV = dt$. Contrary to our usual policy of not including a constant of integration in $V$, here write $V = -(x - t)$ rather than just $V = t$. Observe that the result of the integration by parts is the case $n = 1$ of the formula.

(b) Use induction argument (and integration by parts again) to show that if the formula is valid for $n = k$, then it is also valid for $n = k + 1$.