MA1102
Grunnkurs i analyse II
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Øving 11

1 **a)** Compute two iterations of Newton's method to approximate a zero of the following function

$$f(x) = x^3 + 2x - 1$$

on the interval $[0, 1]$ with the starting point $x_0 = 0$.

b) Use Newton's method to approximate the minimum value of the function

$$f(x) = (x-1)^2 + \cos(x^2),$$

on the interval $[0, 2]$. Use the initial guess $x_0 = 2$. You only need to compute one iteration.

2 **a)** Prove that the sequence $(x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \dots)$ converges for $T(x) = \frac{1}{1+x^2}$ and any $x_0 \in \mathbb{R}$.

b) Prove that the sequence $(x_0, T(x_0), T(T(x_0)), T(T(T(x_0))), \dots)$ converges for $T(x) = \cos(x)$ and any $x_0 \in \mathbb{R}$.

3 Solve the differential equation

$$y'y(1 + x^2) - x = 0$$

4 Verify that the ODE

$$x'(t) = -tx(t), \qquad x(0) = 1$$

satisfies the conditions of the Picard-Lindelöf theorem and compute the next 3 Picard iterations (x_1, x_2, x_3). Start with $x_0(t) = 1$ to match the initial condition.

Can you recognize which Taylor series this approaches? If not that is okay!
Hint: $2 \cdot 4 \cdot 6 \cdots 2n = 2^n \cdot n!$.

Extra:

6 In this exercise we will prove the Picard-Lindelöf theorem which gives conditions under which the ODE

$$x'(t) = f(t, x), \qquad x(t_0) = x_0$$

has a solution in a neighborhood of t_0.

Specifically, assume that f is continuous on

$$R = \{(t,x) : |t - t_0| < a,\ |x - x_0| < b\}$$

and bounded by $c \in \mathbb{R}$. Moreover, we assume that f is *Lipschitz* in its second argument which means that there exists a positive constant k such that

$$|f(t,x) - f(t,y)| \leq k|x - y|$$

for all (t,x) and (t,y) in R.

a) Show that in order to show that the ODE has a solution, it suffices to show that there exists a function $x \in C^1[t_0 - a, t_0 + a]$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Hint: Differentiate; $x \in C^1$ means that x is continuously differentiable.

b) With the interval $J = [t_0 - \beta, t_0 + \beta]$ we can define the closed space

$$X = \left\{ y \in C(J) \ :\ y(t_0) = x_0,\ \sup_{t \in J} |x_0 - y(t)| \leq c\beta \right\}$$

(we assume β is so small that $c\beta < b$). Show that the operator T defined by

$$T(y)(t) = x_0 + \int_{t_0}^{t} f(s, y(s))\, ds, \qquad t \in J$$

maps X to X, i.e., $T(y) \in X$ for $y \in X$.

c) Show that for $y_1, y_2 \in X$,

$$d_\infty(T(y_1), T(y_2)) \leq k\beta d_\infty(y_1, y_2).$$

d) Find a condition on β so that we can use Banach's fixed point theorem to conclude that there exists a unique $x \in C(J)$ such that

$$T(x) = x.$$

e) Combine the above results to conclude that

$$x'(t) = f(t,x), \qquad x(t_0) = x_0$$

has a unique solution on $C^1(J)$.