Norges teknisk–naturvitenskapelige universitet
Department of Mathematical Sciences

MA1102 Grunnkurs i analyse II Vår 2024

Øving 4

Remember to justify all your answers.

1 Let $\{a_j\}$ be a real sequence. Prove that if

$$\liminf\{a_j\} = \limsup\{a_j\},$$

then the sequence $\{a_j\}$ converges. Prove the converse as well.

2 Let $a < b$ be real numbers. Give an example of a sequence whose lim sup is b and whose lim inf is a.

3 For each of the following series determine if they are convergent or divergent.

(a)
$$\frac{1}{3} + \frac{1}{5} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$

(b)
$$\sum_{j=1}^{\infty} \frac{(2^j)^2}{j!}$$

(c)
$$\sum_{n=1}^{\infty} \frac{n^2 + \sqrt{n}}{3n^2 - 4}$$

4 Assume $a_j > 0, b_j > 0, \sum_j a_j$ converges, and $\sum_j b_j$ converges. Does $\sum_j a_j b_j$ converge?

5 One way to define Euler's number, e is by the sum

$$e = \sum_{j=0}^{\infty} \frac{1}{j!}. \tag{1}$$

(a) Prove that the series given above in (1) is convergent.

(b) Using this definition of e given in (1), prove that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Hint for (b): Binomial theorem.