MA1102 Grunnkurs i analyse II Vår 2024

Øving 6

1 For each of the series below determine whether the series converges absolutely, converges conditionally or diverges.

a)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$$

b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^4}$$

2 In the lectures we saw that the infinite union of open sets are open. For this exercise, we prove the corresponding result for closed sets and give counterexamples.

a) Prove that the infinite intersection of closed sets is closed. That is, given a collection $(E_n)_n$ of closed sets, prove that

$$\bigcap_n E_n$$

is closed.

b) Give an example of an infinite collection of closed sets $(E_n)_n$ for which the union $\bigcup_n E_n$ is not closed.

c) Give an example of an infinite collection of open sets $(U_n)_n$ for which the intersection $\bigcap_n U_n$ is not open.

3 Classify whether the following sets are open, closed and/or compact. No motivation is needed.

a)

$$\bigcup_{n\in\mathbb{Z}} \{n\}$$

b)

$$\mathbb{R}$$

c)

$$\mathbb{R} \setminus \mathbb{Q}$$

d)

$$(-1, 1) \cup [2, 3]$$

4 In topology we usually define the **distance** $d(A, B)$ between two the sets A, B as

$$d(A, B) = \inf_{a \in A, b \in B} d(a, b)$$

where $d(a, b)$ is the distance between a and b. Recall that $(a_n)_n$ converges to a if $\lim_{n \to \infty} d(a_n, a) = 0$.

Let K be compact and L closed, and assume that the two sets are disjoint. Show that there is a positive distance between the two sets.
Hint: Assume distance 0 *and derive a contradiction.*

5 Give examples for each of the following situations.

a) An open, unbounded, proper subset of $\mathbb{R}$.

b) Two disjoint open sets with distance zero.

c) A compact set with a finite number of elements.

d) A compact, countably infinite set.