Norwegian University of Science and Technology
Department of Mathematical Sciences

MA1102
Grunnkurs i analyse II
Vår 2024

Exercise set 7

Remember to justify your answers!

1

a) Let $f(x) = \cos(x)$, compute $f^{-1}((0,1))$. That is find the inverse image of the set $(0,1)$ under the function f. Is this set open?

b) Let f be defined as

$$f(x) = \begin{cases} \sin(x) & x < 0, \\ e^{-x} + 1 & x \geq 0. \end{cases}$$

Prove that this function is discontinuous by finding an open set U such that $f^{-1}(U)$ is not open.

c) For $f(x) = e^{-x^2}$, compute $f^{-1}([1,2])$. That is find the inverse image of the interval $[1,2]$ under the function f.

Hint: Recall that given a function $f : D \to \mathbb{R}$ and a subset $W \subset \mathbb{R}$ we define

$$f^{-1}(W) = \{x \in D : f(x) \in W\}.$$

We refer to $f^{-1}(W)$ as *the inverse image of W under f*. For part (b) recall also that we have seen that for a function $f : \mathbb{R} \to \mathbb{R}$ it follows that f is continuous if and only if the inverse image of every open set under f is open. (See also Defintion 5.16 and Theorem 5.17 in Krantz.)

2 The function $f : \mathbb{R} \setminus \{0\} \to [-1,1], x \mapsto \sin(\frac{1}{x})$ is clearly continuous for $x \neq 0$ as it is the composition of two continuous functions. Prove that f cannot be extended to a continuous function on all of $\mathbb{R}$ by showing that the limit

$$\lim_{x \to 0} f(x)$$

does not exist.

3

a) Give an example of a continuous function f and an open set U so that $f(U)$ is not open.

b) Give an example of a discontinuous function f and a closed set E so that $f^{-1}(E)$ is open.

4 If f is continuous on $[0,1]$ and if $f(x)$ is positive for each rational x, then does it follow that f is positive for all x?

5 Let f be a continuous function on a compact set $K \subset \mathbb{R}$. Prove that there exists numbers a and b in K such that $f(a) \leq f(x) \leq f(b)$ for all $x \in K$.

Hint: We have seen in the lectures that the image of a compact set under a continuous function is also compact.

6 In the lectures this week we defined continuity of a function $f : D \to \mathbb{R}$ as follows: We say that f is continuous at the point a in D if for every sequence $\{x_n\}$ in D converging to a it follows that

$$\lim_{x_n \to a} f(x_n) = f(a).$$

Show that this is equivalent to the definition of continuity we saw in MA1101 using $\varepsilon - \delta$. That is f is continuous at the point a in D if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - a| < \delta$ it follows that $|f(x) - f(a)| < \varepsilon$.

Hint: For the second part of the proof, proving that continuity in the sense of sequences implies continuity in the sens of $\varepsilon - \delta$ you might find it helpful to do the proof by contraposition.