



- 1 For each of the following function sequences $(f_n)_n$ and associated domains, determine if the sequence is pointwise convergent to some function f on that domain and if so, is it also uniformly convergent to f ?

a) $I = \mathbb{R}$, $f_n(x) = xe^{-nx^2}$.

b) $I = [-1, 1]$ $f_n(x) = x^{2n+1}$.

c) $I = \mathbb{R}$, $f_n(x) = \frac{1}{n+n(x-n)^2}$.

d) $I = (0, \infty)$, $f_n(x) = \frac{nx}{1+n^2x^2}$.

e) $I = [0, \infty)$, $f_n(x) = ne^{-nx}$.

f) $I = \mathbb{R}$, $f_n(x) = \frac{nx}{n+x}$.

Hint: If $(f_n)_n$ is a sequence of continuous functions which converge uniformly to f , then f must also be continuous.

- 2 Prove that if the function sequence $(f_n)_n$ is uniformly convergent to f on $I \subset \mathbb{R}$, then the sequence also converges pointwise. Also give an example showing that the converse is not true.

- 3 Let $(f_n)_n$ be defined by

$$f_n(x) = \frac{\cos(nx)}{n}.$$

a) Show that $(f_n)_n$ converges uniformly to a function f .

b) Show that $(f'_n)_n$ does **not** converge to f' .

- 4 Define the sequence of functions $(f_n)_{n \geq 2}$ as $f_n : [0, 1] \rightarrow \mathbb{R}$ to be the function

$$f_n(x) = \begin{cases} 0, & x = 0, \\ n, & 0 < x \leq 1/n, \\ 0, & 1/n < x \leq 1. \end{cases}$$

Determine

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx.$$

5 Let $(f_n) : [-1, 1] \rightarrow \mathbb{R}$ be the sequence of functions given by

$$f_n(x) = \frac{n^2}{1 + n^2x^2 + n^2}.$$

This sequence of functions converges to $f(x) = 1/(1 + x^2)$ on $[-1, 1]$. (You do not need to show this). Determine

$$\lim_{n \rightarrow \infty} \int_{-1}^1 f_n(x) dx$$