6.3 ORTOGONAL BASIS

DEF 3.3.1 (NY 6.3.1) EN MENGDE AV VEKTORER $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\} \subseteq \mathbb{R}^n$ KALLES ORTOGONAL HVIS

$$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad \forall i \neq j$$

TM 6.3.1 HVIS $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\} \subseteq \mathbb{R}^n$ ER EN ORTOGONAL MENGDE SLIK AT $\vec{0} \notin S$, SÅ ER S LINEÆRT UAVHENGIG

BEVIS: MÅ VISE AT $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_r\vec{v}_r = \vec{0}$ IMPLISERER $c_1 = c_2 = \dots = c_r = 0$

HVIS $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_r\vec{v}_r = 0 \Rightarrow 0 = \langle \vec{0}, \vec{v}_1 \rangle = \langle c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_r\vec{v}_r, \vec{v}_1 \rangle$

$$= c_1\langle \vec{v}_1, \vec{v}_1 \rangle + c_2\langle \vec{v}_2, \vec{v}_1 \rangle + \dots + c_r\langle \vec{v}_r, \vec{v}_1 \rangle$$

$$= c_1 \|\vec{v}_1\|^2$$

SIDEN $\langle \vec{v}_i, \vec{v}_j \rangle = 0 \ \forall i \neq j$

$$\overset{\vec{v}_1 \neq \vec{0}}{\Longrightarrow} c_1 = 0$$

PÅ SAMME MÅTEN: $0 = \langle \vec{0}, \vec{v}_i \rangle = c_i \langle \vec{v}_i, \vec{v}_i \rangle \Rightarrow c_i = 0 \ \forall i$

$\Rightarrow c_i = 0 \ \forall i$

$\square$

ORTOGONAL BASIS: BASIS SOM ER EN ORTOGONAL MENGDE

TM 6.3.5: LA W VÆRE ET UNDERROM AV $\mathbb{R}^n$ SLIK AT $W \neq \{\vec{0}\}$, SÅ GJELDER W HAR EN ORTOGONAL BASIS

BEVIS: LA $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ VÆRE EN BASIS FOR W

MÅL: KONSTRUERE EN ORTOGONAL BASIS BASERT PÅ $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$

$\rightsquigarrow$ GRAM-SCHMIDT PROSESSEN

1) LA $\vec{u}_1 = \vec{v}_1$ $\Rightarrow$ $\left.\begin{array}{l} \cdot)\ \{\vec{u}_1, \vec{v}_2, \dots, \vec{v}_r\} \text{ LIN UAVH} \\ \cdot)\ \text{span}\{\vec{u}_1, \vec{v}_2, \dots, \vec{v}_r\} = W \end{array}\right\} \Rightarrow \{\vec{u}_1, \vec{v}_2, \dots, \vec{v}_r\}$ BASIS

2) LA $\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} u_1$ $\overset{3.3.2}{\Longrightarrow}$ $\left.\begin{array}{l} \cdot)\ \langle \vec{u}_2, \vec{u}_1 \rangle = 0 \\ \cdot)\ v_2 \ldots \text{LIN KOMB AV } \vec{u}_1 \text{ OG } u_2 \end{array}\right\}$

$\overset{4.3.2}{\underset{4.6.4}{\Longrightarrow}}$ $\left.\begin{array}{l} \cdot)\ \langle \vec{u}_2, \vec{u}_1 \rangle = 0 \\ \cdot)\ \text{span}\{\vec{u}_1, \vec{u}_2, \vec{v}_3, \dots, \vec{v}_r\} = W \\ \cdot)\ \{\vec{u}_1, \vec{u}_2, \vec{v}_3, \dots, \vec{v}_r\} \text{ LIN UAVH} \end{array}\right\} \Rightarrow \begin{array}{l} \cdot)\ \langle \vec{u}_2, \vec{u}_1 \rangle = 0 \\ \cdot)\ \{\vec{u}_1, \vec{u}_2, \vec{v}_3, \dots, \vec{v}_r\} \text{ BASIS} \end{array}$

3) LA $\vec{u}_3=\vec{v}_3-\dfrac{\langle\vec{v}_3,\vec{u}_2\rangle}{\|\vec{u}_2\|^2}\vec{u}_2-\dfrac{\langle\vec{v}_3,\vec{u}_1\rangle}{\|\vec{u}_1\|^2}\vec{u}_1 \quad\Longrightarrow\quad$ ·) $\langle\vec{u}_2,\vec{u}_3\rangle=0=\langle\vec{u}_1,\vec{u}_3\rangle$ ·) $\vec{v}_3$ LIN KOMB AV $\vec{u}_1,\vec{u}_2,\vec{u}_3$

$\overset{4.3.2}{\underset{4.6.4}{\Longrightarrow}}$
·) $\langle\vec{u}_1,\vec{u}_2\rangle=\langle\vec{u}_1,\vec{u}_3\rangle=\langle\vec{u}_2,\vec{u}_3\rangle=0$
·) span$\{\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{v}_4,\dots,\vec{v}_r\}=W$
·) $\{\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{v}_4,\dots,\vec{v}_r\}$ LIN UAVH

$\Longrightarrow$
·) $\langle\vec{u}_i,\vec{u}_j\rangle=0 \quad \forall i\neq j \quad 1\le i,j\le 3$
·) $\{\vec{u}_1,\vec{u}_2,\vec{u}_3,\vec{v}_4,\dots,\vec{v}_r\}$ BASIS

$\vdots$

i) LA $\vec{u}_i=\vec{v}_i-\dfrac{\langle\vec{v}_i,\vec{u}_{i-1}\rangle}{\|\vec{u}_{i-1}\|^2}\vec{u}_{i-1}-\dots-\dfrac{\langle\vec{v}_i,\vec{u}_2\rangle}{\|\vec{u}_2\|^2}\vec{u}_2-\dfrac{\langle\vec{v}_i,\vec{u}_1\rangle}{\|\vec{u}_1\|^2}\vec{u}_1$

$\Longrightarrow$
·) $\langle\vec{u}_i,\vec{u}_j\rangle=0 \quad \forall j<i$
·) $\vec{v}_i$ LIN. KOMB AV $\vec{u}_1,\vec{u}_2,\dots,\vec{u}_i$

$\overset{4.3.2}{\underset{4.6.4}{\Longrightarrow}}$
·) $\langle\vec{u}_k,\vec{u}_j\rangle=0 \quad \forall k\neq j \quad 1\le k,j\le i$
·) $\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_i,\vec{v}_{i+1},\dots,\vec{v}_r\}$ BASIS

ETTER r STEG: $\{\vec{u}_1,\vec{u}_2,\dots,\vec{u}_r\}$ .. ORTOGONAL BASIS TIL W

EKS: GITT $\vec{v}_1=(1,1,1)$, $\vec{v}_2=(0,1,1)$, $\vec{v}_1=(0,0,1)$
ANVENDT GRAM-SCHMIDT FOR Å FINNE EN ORTOGONAL BASIS FOR $\mathbb{R}^3$

$\vec{u}_1=\vec{v}_1=(1,1,1)$

$\vec{u}_2=\vec{v}_2-\dfrac{\langle\vec{v}_2,\vec{u}_1\rangle}{\|\vec{u}_1\|^2}\vec{u}_1=\vec{v}_2-\dfrac{2}{3}\vec{u}_1=(0,1,1)-\dfrac{2}{3}(1,1,1)=\left(-\dfrac{2}{3},\dfrac{1}{3},\dfrac{1}{3}\right)$

$\vec{u}_3=\vec{v}_3-\dfrac{\langle\vec{v}_3,\vec{u}_2\rangle}{\|\vec{u}_2\|^2}\vec{u}_2-\dfrac{\langle\vec{v}_3,\vec{u}_1\rangle}{\|\vec{u}_1\|^2}\vec{u}_1=\vec{v}_1-\dfrac{\frac{1}{3}}{\frac{6}{9}}\vec{u}_2-\dfrac{1}{3}\vec{u}_1$

$=(0,0,1)-\dfrac{1}{2}\left(-\dfrac{2}{3},\dfrac{1}{3},\dfrac{1}{3}\right)-\dfrac{1}{3}(1,1,1)$

$=\left(0,-\dfrac{1}{2},\dfrac{1}{2}\right)$

$\Rightarrow \left\{(1,1,1),\left(-\dfrac{2}{3},\dfrac{1}{3},\dfrac{1}{3}\right),\left(0,-\dfrac{1}{2},\dfrac{1}{2}\right)\right\}$ ... ORTOGONAL BASIS FOR $\mathbb{R}^3$

OBS EN FØLGE AV GRAM-SCHMIDT:

LA $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\} \subseteq \mathbb{R}^n$ VÆRE EN ORTOGONAL BASIS FOR W (ET UNDERROM AV $\mathbb{R}^n$)

LA $\vec{v} \in \mathbb{R}^n$, SÅ KAN $\vec{v}$ ENTYDIG SKRIVES SOM

$$\vec{v} = \vec{w} + \vec{w}^{\perp}$$

DER

- $\vec{w} = \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2 + \dots + \frac{\langle \vec{v}, \vec{u}_r \rangle}{\|\vec{u}_r\|^2} u_r \in W$

  $\vec{w} = \mathrm{proj}_W(\vec{v})$ ... PROJEKSJON AV VEKTOREN $\vec{v}$ PÅ UNDERROMMET W

- $\vec{w}^{\perp} \notin W$ OG $\langle \vec{w}^{\perp}, \vec{u} \rangle = 0 \ \forall \vec{u} \in W$

I TILFELLER DER $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ ER EN BASIS FOR W (IKKE NØDVENDIGVIS ORTOGONAL) ANVENDER MAN FØRST GRAM SCHMIDT PÅ $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ FOR Å FÅ EN ORTOGONAL BASIS $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$, FOR SÅ Å FINNE $\mathrm{proj}_W(\vec{v})$ FOR $\vec{v} \in \mathbb{R}^n$

EKS: LA $W = \mathrm{span}\{\vec{v}_1, \vec{v}_2\}$ DER $\vec{v}_1 = (1,2,3)$, $\vec{v}_2 = (2,3,4)$.

GITT $\vec{v} = (3,5,3)$. FINN $\mathrm{proj}_W(\vec{v})$.

- FINN EN ORTOGONALBASIS $\{\vec{u}_1, \vec{u}_2\}$ TIL W

$$\vec{u}_1 = \vec{v}_1 = (1,2,3)$$

$$\vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 = (2,3,4) - \frac{20}{14}(1,2,3) = (2,3,4) - \frac{10}{7}(1,2,3) = \left(\frac{4}{7}, \frac{1}{7}, -\frac{2}{7}\right)$$

$$\{\vec{u}_1, \vec{u}_2\} = \left\{(1,2,3), \left(\frac{4}{7}, \frac{1}{7}, -\frac{2}{7}\right)\right\}$$

- FINN $\mathrm{proj}_W(\vec{v})$

$$\mathrm{proj}_W(\vec{v}) = \frac{\langle \vec{v}, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle \vec{v}, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2$$

$$= \frac{22}{14}(1,2,3) + \frac{\frac{11}{7}}{\frac{21}{49}} \left(\frac{4}{7}, \frac{1}{7}, -\frac{2}{7}\right)$$

$$= \frac{11}{7}(1,2,3) + \frac{11}{21}(4,1,-2)$$

$$= \frac{11}{21}(7,7,7) = \frac{11}{3}(1,1,1)$$

$$\left[W^{\perp} = (3,5,3) - \frac{11}{3}(1,1,1) = \frac{1}{3}\left[(9,15,9) - (11,11,11)\right] = \frac{1}{3}(-2,4,-2) = \left(-\frac{2}{3}, \frac{4}{3}, -\frac{2}{3}\right) \right]$$