Eksamen juni 2005

① $t \in R$ $\qquad A_t = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & t \\ 0 & t & 0 & 1 \end{bmatrix}$

a) Undersøker når $\det(A_t) = 0$.

$$\det(A_t) = t \begin{vmatrix} 1 & t & 0 \\ 0 & 1 & t \\ t & 0 & 1 \end{vmatrix} = t\left(1 \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} - t \begin{vmatrix} 0 & t \\ t & 1 \end{vmatrix}\right)$$

$$= t + t^4 = t(1+t^3)$$

$$\det(A_t) = 0 \Leftrightarrow t(1+t^3) = 0$$

$$t = 0 \lor t = -1$$

Da er $\operatorname{rang}(A_t) = 4$ når $t \neq 0 \land t \neq -1$.

Undersøker A_0, A_{-1}:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \operatorname{rang}(A_0) = 3$$

$$A_{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \operatorname{rang}(A_{-1}) = 3$$

$$\text{så} \quad \operatorname{rang}(A_t) = \begin{cases} 4 & \text{når } t \neq 0 \land t \neq -1 \\ 3 & \text{når } t = 0 \lor t = -1 \end{cases}$$

b) 1. Nøyaktig en løsning: når $\text{rang}(A_t) = 4$,
dvs. $t \neq 0 \wedge t \neq -1$.

Setter opp $\overbrace{\text{systemet}}^{\text{totalmatrisen til}}$ for $t=0$ og $t=-1$:

$$\tilde{A}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

her er $\text{rang}(A_0) = 3 < \text{rang}(\tilde{A}_0) = 4$, så
for $t=0$ har systemet ingen løsning.

$$\tilde{A}_{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

her er $\text{rang}(A_{-1}) = 3 = \text{rang}(\tilde{A}_{-1}) < 4$ (# ukjente), så
for $t=-1$ har systemet uendelig mange løsninger.

1) $t \neq 0 \wedge t \neq -1$

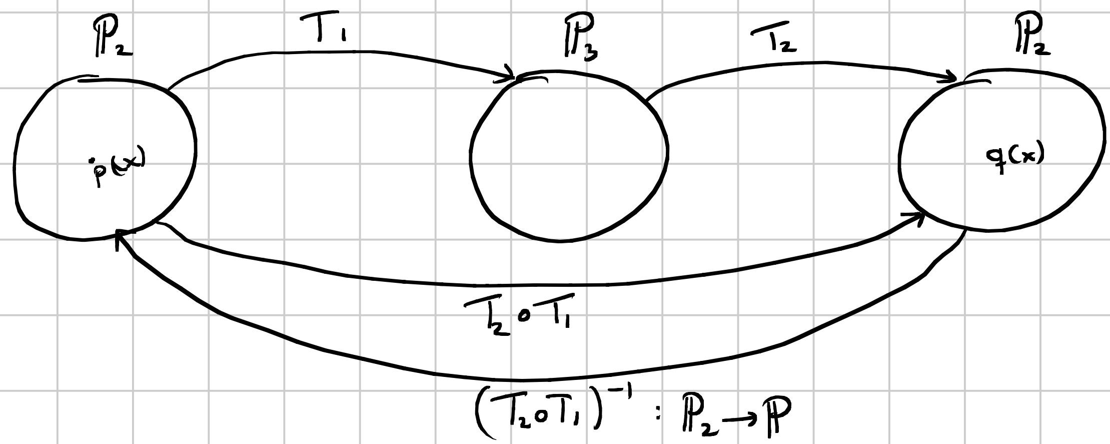
2) $t = 0$

3) $t = -1$

(2) $p_1(x) = a_0 + a_1x + a_2x^2$ $\qquad$ $p_2(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$T_1 : \mathbb{P}_2 \to \mathbb{P}_3$, $T_1(p) = xp(x)$, $T_1(p_1) = a_0x + a_1x^2 + a_2x^3$

$T_2 : \mathbb{P}_3 \to \mathbb{P}_2$, $T_2(p) = \dfrac{d\,p(x)}{dx}$, $T_2(p_2) = a_1 + 2a_2x + 3a_3x^2$



$$(T_2 \circ T_1)(p) = T_2(T_1(p)) = T_2(a_0x + a_1x^2 + a_2x^3)$$
$$= a_0 + 2a_1x + 3a_2x^2$$

Gitt

$q(x) = b_0 + b_1x + b_2x^2 \in \mathbb{P}_2$ vil vi finne $p(x) = a_0 + a_1x + a_2x^2$

$$p(x) = (T_2 \circ T_1)^{-1}(q(x))$$
$$\Updownarrow$$
$$(T_2 \circ T_1)p = q$$
$$a_0 + 2a_1x + 3a_2x^2 = b_0 + b_1x + b_2x^2$$
$$\Updownarrow$$

$$\begin{aligned} a_0 &= b_0 \\ 2a_1 &= b_1 \\ 3a_2 &= b_2 \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} a_0 &= b_0 \\ a_1 &= \tfrac{1}{2}b_1 \\ a_2 &= \tfrac{1}{3}b_2 \end{aligned}$$

dvs. $(T_2 \circ T_1)^{-1}(q) = (T_2 \circ T_1)^{-1}(b_0 + b_1x + b_2x^2) = p(x)$

$$= \underline{\underline{b_0 + \frac{b_1}{2}x + \frac{b_2}{3}x^2}}$$

b) $B = \{1, (x-1), (x-1)^2\}$ $\leftarrow$

Skal finne $[T_2 \circ T_1]_B$.

dvs.

$$\Big[\, [(T_2\circ T_1)(1)]_B \quad [(T_2\circ T_1)(x-1)]_B \quad [(T_2\circ T_1)((x-1)^2)]_B \,\Big]$$

$$(T_2\circ T_1)(1) = T_2(T_1(1)) = 1 \qquad [(T_2\circ T_1)(1)]_B = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$(T_2\circ T_1)(x-1) = T_2(\underbrace{T_1(x-1)}^{x^2-x}) = \underbrace{2x-1}_{2(x-1)+1} \qquad [(T_2\circ T_1)(x-1)]_B = \begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$c_1\cdot 1 + c_2\cdot(x-1) + c_3\cdot \overbrace{(x-1)^2}^{x^2-2x+1} = 2x-1$$

$$\begin{array}{ccc} V_1 & V_2 & V_3 \end{array}$$

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$$

$\uparrow$ $\{1, x, x^2\}$

$$\begin{pmatrix}1 & -1 & 1\\0 & 1 & -2\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$$

$$\begin{aligned}(T_2\circ T_1)((x-1)^2) &= T_2(T_1((x-1)^2))\\ &= T_2(T_1(x^2-2x+1)) = T_2(x^3-2x^2+x)\\ &= 3x^2-4x+1\\ &= 3(x-1)^2 + 2(x-1) + 0\cdot 1\end{aligned}$$

$$\underbrace{3x^2-6x+3+2x-2}_{3x^2-4x+1}$$

$$[(T_2\circ T_1)((x-1)^2)]_B = \begin{pmatrix}0\\2\\3\end{pmatrix} \qquad [T_2\circ T_1]_B = \begin{bmatrix}1 & 1 & 0\\0 & 2 & 2\\0 & 0 & 3\end{bmatrix}$$

c) Skal finne rang og nullitet til $T_2 : \mathbb{P}_3 \to \mathbb{P}_2$

Rang-teoremet: $\text{rang}(T_2) + \text{nullity}(T_2) = \dim(\mathbb{P}_3)$

$$\dim(R(T_2)) + \dim(\ker(T_2)) = 4$$

$$\ker(T_2) = \{q \in \mathbb{P}_3 : T_2(q) = 0\} = \{q \in \mathbb{P}_3 : \tfrac{dq}{dx} = 0\}$$

$$= \{\text{konstanter}\} = \text{span}\{1\}$$

så $\dim(\ker(T_2)) = \underline{\underline{1 = \text{nullity}(T_2)}}$

Da er $\text{rang}(T_2) = 4 - \text{nullity}(T_2) = 4 - 1 = \underline{\underline{3}}$

(5) $A \in M_{nn}$, og $A^* = -A$. $A^* = \bar{A}^T$

Skal vise at egenverdiene til A er rene imaginære tall.

$Ax = \lambda x$, anta at $x^* x = 1$

$$x^* A x = x^* \lambda x = \lambda x^* x = \lambda I_1 = \lambda \leftarrow \text{tall}$$

$$\lambda^* = \bar{\lambda}$$

$$\lambda^* = (x^* A x)^* = x^* A^* x^{**} = x^* A^* x \overset{A^* = -A}{=} -x^* A x = -\lambda$$

Har vist: $\bar{\lambda} = -\lambda$, $\left.\begin{array}{l}\lambda = a + ib \\ \bar{\lambda} = a - ib\end{array}\right\}$ $\begin{array}{l}\bar{\lambda} = -\lambda \\ a - ib = -(a+ib) \\ a - ib = -a - ib \\ 2a = 0\end{array}$

Så $\underline{\underline{\lambda = ib}}$, rent imaginært. $(i = \sqrt{-1})$

Oppgave 5 nov. 96

1. Skal vise at $T: \mathbb{P}_2 \to P_2$ gitt ved $T(f) = f + f' + f''$ er en linear transformasjon.

(*) $T(f+g) = T(f) + T(g)$

(**) $T(kf) = kT(f)$

(*): $T(f+g) = (f+g) + (f+g)' + (f+g)'' = f + g + f' + g' + f'' + g''$

$$= \underbrace{(f + f' + f'')}_{} + \underbrace{(g + g' + g'')}_{}$$

$$= T(f) \qquad + \quad T(g)$$

(**) $T(kf) = (kf) + (kf)' + (kf)'' = k(f + f' + f'') = kT(f)$