NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for
MA1202/MA6202 Linear Algebra with Applications
Solutions

Academic contact during examination: Silvius Klein[a], Magnus B. Landstad[b]

Phone:

Examination date: May 26, 2015

Examination time (from–to): 09:00–13:00

Permitted examination support material: No printed or hand-written support material is allowed. A specific basic calculator is allowed.

Language: English

Number of pages: 14

Number pages enclosed: 0

Checked by:

Date Signature

Note: These solutions are very detailed for *pedagogical* reasons. Students are not required to provide the same level of detail, but reasons for all answers are required.

Problem 1 This problem involves the concept of *linear independence.*

a) Write the definition of a linearly independent set of vectors in an abstract vector space.

Solution. Let V be a vector space. A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is *linearly independent* if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_r\mathbf{v}_r = \mathbf{0} \tag{1a}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_r = 0.$$

□

b) Consider the vector space $\mathbf{P}_2$ of polynomials of degree ≤ 2.
Determine if the following sets are linearly independent in $\mathbf{P}_2$:

$$\begin{array}{ll} (i) & \{1 - x, 1 + x, x^2\}. \\ (ii) & \{1 + x, 1 + x^2, x - x^2\}. \end{array}$$

Solution. (i) Consider the vector equation

$$\begin{aligned} c_1(1 - x) + c_2(1 + x) + c_3x^2 &= 0 \\ (c_1 + c_2) + (-c_1 + c_2)\,x + c_3\,x^2 &= 0. \end{aligned}$$

We then get

$$c_1 + c_2 = 0, -c_1 + c_2 = 0, c_3 = 0,$$

which implies that

$$c_1 = 0, c_2 = 0, c_3 = 0,$$

hence the set $\{1 - x, 1 + x, x^2\}$ is linearly independent in $\mathbf{P}_2$.

(ii) Note that $(1+x)-(1+x^2) = x-x^2$, which shows that $\{1+x, 1+x^2, x-x^2\}$ is *not* linearly independent in $\mathbf{P}_2$.

$\square$

c) Let V be an *inner product space*. Prove that if two non-zero vectors in V are orthogonal, then they are linearly independent as well.

Solution. Let $\mathbf{v}_1, \mathbf{v}_2$ be two non-zero vectors in V such that

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0.$$

Consider now the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}. \tag{1c}$$

We need to show that $c_1 = 0$ and $c_2 = 0$.

Take the inner product with $\mathbf{v}_1$ on both sides of the equation (1c):

$$\langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \mathbf{v}_1 \rangle = \langle \mathbf{0}, \mathbf{v}_1 \rangle .$$

Compute the two sides of the equation above:

$$\begin{aligned} \text{Right hand side } &= 0, \\ \text{Left hand side } &= c_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_1 \rangle \\ &= c_1\|\mathbf{v}_1\|^2 + c_2 \cdot 0 \\ &= c_1\|\mathbf{v}_1\|^2. \end{aligned}$$

Since the two sides are equal, we conclude that

$$c_1\|\mathbf{v}_1\|^2 = 0\,.$$

But $\mathbf{v}_1 \neq \mathbf{0}$, so $\|\mathbf{v}_1\| \neq 0$, which shows that $c_1 = 0$.

Using equation (1c) again, it follows that

$$c_2\mathbf{v}_2 = \mathbf{0},$$

and since $\mathbf{v}_2 \neq \mathbf{0}$, we must have that $c_2 = 0$.

$\square$

d) Prove that $\sin x$ and $\cos x$ are linearly independent in $\mathbf{F}([0,\pi])$ = the space of all functions $f\colon [0,\pi] \to \mathbb{R}$.

Solution. We may argue in two different ways.

1. Directly: Consider the "vector" equation:

$$c_1 \sin x + c_2 \cos x = \mathbf{0},$$

which must hold for *all* values of x in $[0,\pi]$.

Choosing some particular values for x, we will conclude that $c_1 = 0$ and $c_2 = 0$.

Let $x = 0$. Since $\sin 0 = 0$ and $\cos 0 = 1$, we get $c_1 \cdot 0 + c_2 \cdot 1 = 0$, so $c_2 = 0$.

Let $x = \frac{\pi}{2}$. Since $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$, we get $c_1 \cdot 1 + c_2 \cdot 0 = 0$, so $c_1 = 0$.

2. Using inner products: Consider the subspace $\mathbf{C}([0,\pi])$ of *continuous* functions on $[0,\pi]$. This subspace contains our given functions $\sin x$ and $\cos x$.

We already know that $\mathbf{C}([0,\pi])$ is an *inner product space* with the operation

$$\langle f(x), g(x)\rangle = \int_0^\pi f(x)\cdot g(x)\,dx.$$

Compute

$$\begin{aligned}\langle \sin x, \cos x\rangle &= \int_0^\pi \sin x \cdot \cos x \, dx\\ &= \int_0^\pi \frac{1}{2}\sin 2x\,dx\\ &= -\frac{1}{2}\,\frac{\cos 2x}{2}\Big|_0^\pi = 0.\end{aligned}$$

This shows that the "vectors" $\sin x$ and $\cos x$ are *orthogonal.* By question 1c), they must be linearly independent in $\mathbf{C}([0,\pi])$, so also in $\mathbf{F}([0,\pi])$. □

Problem 2 Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

and let $V = \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

a) Are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent?
Find a basis for V.

Solution. We consider the matrix

$$A = \left[\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3|\mathbf{v}_4\right] = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 2 & 1 & -2 & 1 \\ 0 & -1 & 4 & 1 \end{bmatrix},$$

whose columns are the given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$.

Since

$$\text{column space of } A = \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = V,$$

we will find a basis for V by finding a basis for the column space of A.

We use Gaussian elimination to find the reduced row echelon form (RREF) of A:

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 2 & 1 & -2 & 1 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R_2+R_4} \begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -6 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_3}$$

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & -10 & -5 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & -6 & -4 \end{bmatrix} \xrightarrow{6R_3+R_4} \begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & -10 & -5 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the reduced echelon form (REF).

We perform one more step to obtain the RREF:

$$\begin{bmatrix} \boxed{1} & 0 & 4 & 3 \\ 0 & \boxed{1} & -10 & -5 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[10R_3+R_2]{-4R_3+R_1} \begin{bmatrix} \boxed{1} & 0 & 0 & 1/3 \\ 0 & \boxed{1} & 0 & 5/3 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the RREF of A is

$$R = \begin{bmatrix} \boxed{1} & 0 & 0 & 1/3 \\ 0 & \boxed{1} & 0 & 5/3 \\ 0 & 0 & \boxed{1} & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\mathbf{c}_1 | \mathbf{c}_2 | \mathbf{c}_3 | \mathbf{c}_4\right].$$

Note the following relation between the columns of R:

$$\mathbf{c}_4 = \frac{1}{3}\mathbf{c}_1 + \frac{5}{3}\mathbf{c}_2 + \frac{2}{3}\mathbf{c}_3.$$

The same relation will hold between the columns of the original matrix A, in other words:

$$\mathbf{v}_4 = \frac{1}{3}\mathbf{v}_1 + \frac{5}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3,$$

showing that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are *not* linearly independent.

There are, of course, other arguments for why these vectors are not linearly independent, for instance because the determinant of A is 0 (compute it!).

Now regarding the basis for $V = \text{ span } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

The columns with leading 1's in R are the 1st, 2nd and 3rd, so the *corresponding* 1st, 2nd and 3rd columns of A form a basis for the column space of A.

In conclusion: a basis for V is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. □

b) Find an orthogonal basis for V.

Solution. We apply the Gram-Schmidt process to the basis $\mathcal{B}$ determined above, to obtain an *orthogonal* basis $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

Step 1:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1.$$

We compute separately

$$\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\rangle = 2.$$

$$\|\mathbf{w}_1\|^2 = \|\mathbf{v}_1\|^2 = 1^2 + 2^2 = 5.$$

Then

$$\mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix}$$

Step 2:

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2} \mathbf{w}_2.$$

We compute separately

$$\langle \mathbf{v}_3, \mathbf{w}_1 \rangle = \left\langle \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\rangle = 4 - 4 = 0.$$

$$\langle \mathbf{v}_3, \mathbf{w}_2 \rangle = \left\langle \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix} \right\rangle = -8/5 - 2/5 - 4 = -6.$$

$$\|\mathbf{w}_2\|^2 = 4/25 + 1/25 + 1 = 6/5.$$

Then

$$\mathbf{w}_3 = \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix} - 0 - \frac{-6}{6/5} \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix}.$$

Therefore, an orthogonal basis for V is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

□

c) Does there exist a non-zero vector $\mathbf{u}$ in $\mathbb{R}^4$ which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

Solution. In question 1b) we have obtained an orthogonal basis $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the subspace V of $\mathbb{R}^4$. Since $\dim(V) = 3$ and $\dim(\mathbb{R}^4) = 4$, $\mathcal{B}'$ can be *enlarged* to an orthogonal basis for $\mathbb{R}^4$, by adding one extra vector $\mathbf{u}$.

As an element of a basis, $\mathbf{u}$ must be non-zero.

As an element of an orthogonal basis in $\mathbb{R}^4$, $\mathbf{u}$ must be orthogonal to all other elements $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of that basis. Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for the subspace V, $\mathbf{u}$ is in fact orthogonal to *all* vectors in this subspace, and in particular it must be orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, since they are elements of V.

Therefore, to answer to the question is *yes*, such a vector exists. □

Problem 3

This problem involves *orthogonal diagonalization* of a matrix. Let

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

a) Explain why the matrix A must be orthogonally diagonalizable.

Solution. The matrix A is *symmetric*, hence it is orthogonally diagonalizable. □

b) Find the eigenvalues of A and bases for the corresponding eigenspaces.

Solution. We first compute the characteristic polynomial of A:

$$\begin{aligned} p_A(\lambda) = \det(\lambda I - A) &= \det \begin{bmatrix} \lambda - 3 & 0 & -1 \\ 0 & \boxed{\lambda - 2} & 0 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \\ &= (\lambda - 2) \cdot \det \begin{bmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{bmatrix} \\ &= (\lambda - 2)\left((\lambda - 3)^2 - 1\right) = (\lambda - 2)(\lambda - 2)(\lambda - 4). \end{aligned}$$

The *eigenvalues* of A are the solutions to $p_A(\lambda) = 0$:

$$\begin{aligned} &\lambda = 2 \text{ (with algebraic multiplicity 2)}, \\ &\lambda = 4 \text{ (with algebraic multiplicity 1)}. \end{aligned}$$

We now compute a basis for the *eigenspace* corresponding to each eigenvalue. These eigenspaces are the *null spaces* of $\lambda I - A$.

For $\boxed{\lambda = 2}$ we get $\lambda I - A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$. The corresponding system is

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

leading to just

$$-x_1 - x_3 = 0,$$

as the 2nd equation is $0 = 0$ and the 3rd is the same as the first.

The general solution to the system is then

$$x_1 = t,\ x_2 = s,\ x_3 = -t \quad \text{for any } t, s.$$

Choose first $t = 1, s = 0$ and then $t = 0, s = 1$.

Therefore, a basis for the eigenspace corresponding to $\lambda = 2$ is $\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\boxed{\lambda = 4}$ we get $\lambda I - A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. The corresponding system is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

leading to

$$\begin{cases} x_1 \quad\quad -x_3 = 0 & \to \quad x_1 = x_3 \\ \quad 2x_2 \quad\quad = 0 & \to \quad x_2 = 0 \\ -x_1 \quad\quad +x_3 = 0 & \to \quad x_1 = x_3. \end{cases}$$

The general solution to the system is then

$$x_1 = t,\ x_2 = 0,\ x_3 = t \quad \text{for any } t.$$

Take $t = 1$. A basis for the eigenspace corresponding to $\lambda = 4$ is then $\{\mathbf{u}_3\}$, where

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\square$

c) Find an *orthonormal* basis for each eigenspace of A.

Solution. The vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are already orthogonal: $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$. We only need to normalize them to have norm 1.

We have:

$$\|\mathbf{u}_1\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}, \quad \|\mathbf{u}_2\| = 1,$$

so for the eigenspace corresponding to $\lambda = 2$, we obtain the *orthonormal* basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now we normalize the vector $\mathbf{u}_3$. We have $\|\mathbf{u}_3\| = \sqrt{1^2 + 1^2} = \sqrt{2}$, so for the eigenspace corresponding to $\lambda = 4$, we obtain the *orthonormal* basis $\{\mathbf{v}_3\}$, where

$$\mathbf{v}_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}.$$

□

d) Find matrices P and D such that $P^T A P = D$.

Solution. We have

$$P = \begin{bmatrix} \mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

□

Problem 4 This problem involves the concept of *linear transformations.*

Let $T\colon \mathbf{P}_2 \to \mathbf{P}_2$ be the linear operator defined by

$$T(p(x)) = p(x+2).$$

Let $B = \{1, x, x^2\}$ be the standard basis in $\mathbf{P}_2$.

a) Find the matrix $[T]_{B,B}$ (the matrix for T relative to the standard basis). Explain the relationship between $[T]_{B,B}$ and T using a diagram.

Solution. We know that in general, if $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is a basis in a vector space V, and if $T\colon V \to V$ is a linear operator, then

$$[T]_{B,B} = \Big[\, [T(\mathbf{u}_1)]_B \,|\, [T(\mathbf{u}_2)]_B \,|\, \ldots \,|\, [T(\mathbf{u}_n)]_B \Big].$$

Since $B = \{1, x, x^2\}$, using the given formula $T(p(x)) = p(x+2)$, we obtain

$$\begin{array}{rcccl} T(1) & = & 1 & = & 1 + 0\,x + 0\,x^2 \\ T(x) & = & x+2 & = & 2 + 1\,x + 0\,x^2 \\ T(x^2) & = & (x+2)^2 & = & 4 + 4\,x + 1\,x^2. \end{array}$$

Therefore,

$$[T(1)]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_B = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^2)]_B = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix},$$

so

$$[T]_{B,B} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

The following diagram shows that the linear transformation $T\colon \mathbf{P}_2 \to \mathbf{P}_2$ may be regarded as the multiplication by the matrix $[T]_{B,B}$ seen as a linear map : $\mathbb{R}^3 \to \mathbb{R}^3$.

$$\begin{array}{ccc} p(x) \in \mathbf{P}_2 & \xrightarrow{\text{Linear transf. } T} & T(p(x)) \in \mathbf{P}_2 \\ \Big\downarrow \text{isomorphism} & & \Big\downarrow \text{isomorphism} \\ [p(x)]_B \in \mathbb{R}^3 & \xrightarrow{\text{Multip. by } [T]_{B,B}} & [T(p(x))]_B \in \mathbb{R}^3. \end{array}$$

Formally, the above diagram means

$$[T(p(x))]_B = [T]_{B,B} \cdot [p(x)]_B.$$

□

b) Find $\ker(T)$.

Solution. There are several ways to answer this.

1. Directly: $\ker(T)$ = set of all polynomials $p(x)$ in $\mathbf{P}_2$ so that $T(p(x)) = 0$.

But $T(p(x)) = p(x+2)$ and if $p(x+2) = 0$ for all x, then $p(x) = 0$.

We conclude that $\ker(T) = \{0\}$, meaning the zero subspace of $\mathbf{P}_2$.

2. Via $[T]_{B,B}$: In question 4a) we obtained

$$[T]_{B,B} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is an *upper triangular* matrix, hence its determinant is the product of the diagonal entries, which equals $1 \neq 0$.

From this we conclude that the matrix for T in the basis B is *invertible*, which already says that T is an *isomorphism*. In particular T must be one to one (hence also $\ker(T) = \{0\}$), and T must be onto.

Thus, the matrix $[T]_{B,B}$ helped us answer all the questions, including 1c) below.

□

c) Is T one to one? Is T onto? Is T an isomorphism? Justify your answers.

Solution. We may answer this another way. Knowing that $\ker(T) = \{0\}$, and given that T is a linear operator on a *finite dimensional* vector space, by a theorem studied in class, it must be one to one, onto and hence an isomorphism as well. □

Problem 5

Tom is in either of the following two states: happy or sad. It has been observed that if he is happy one day, then the likelihood of being happy the next day is $\frac{4}{5}$, while if he is sad one day, the likelihood of being sad the next day is $\frac{1}{3}$.

a) Write the transition matrix for the Markov process described above.

Solution. The transition matrix of a Markov process encodes the probability of transitioning from a given current state to the next state.

$$P = \begin{array}{c} \begin{matrix} \text{happy} & \text{sad} \end{matrix} \\ \begin{bmatrix} \frac{4}{5} & \frac{2}{3} \\ \\ \frac{1}{5} & \frac{1}{3} \end{bmatrix} \end{array} \begin{matrix} \text{happy} \\ \\ \text{sad} \end{matrix}$$

□

b) Find the steady state vector of this process.

Solution. P is a *regular* stochastic matrix, since its entries are all positive, and its columns are probability vectors.

Then its steady state vector $\mathbf{q}$ exists, and it is the unique solution to the equation $P\,\mathbf{q} = \mathbf{q}$, for which the sum of its components is 1.

Let us solve it. The equation $P\,\mathbf{q} = \mathbf{q}$ is equivalent to $(I - P)\,\mathbf{q} = \mathbf{0}$.

$$I - P = \begin{bmatrix} \frac{1}{5} & -\frac{2}{3} \\ \\ -\frac{1}{5} & \frac{2}{3} \end{bmatrix}.$$

Then

$$\begin{bmatrix} \frac{1}{5} & -\frac{2}{3} \\ \\ -\frac{1}{5} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leads to $\frac{1}{5}\,q_1 - \frac{2}{3}\,q_2 = 0$ (the second equation is the same).

We also need to have that $q_1 + q_2 = 1$ (as the components of $\mathbf{q}$ must add up to 1).

We obtain the system

$$\begin{cases} \frac{1}{5}q_1 - \frac{2}{3}q_2 = 0 \\ q_1 + q_2 = 1. \end{cases}$$

We solve it and get $q_1 = \frac{10}{13}$ and $q_2 = \frac{3}{13}$. Therefore, the steady state vector of this Markov process is

$$\mathbf{q} = \begin{bmatrix} \frac{10}{13} \\ \\ \frac{3}{13} \end{bmatrix}.$$

□

c) Explain the meaning of this vector and indicate what are the chances that *on the long run*, Tom is happy on any given day.

Solution. The steady state vector of a Markov process encodes the long term probability of each state, and it is independent of the current state.

Therefore, regardless of Tom's current state of happiness, on the long run, the likelihood of him being happy on any given day is $\frac{10}{13} \approx 0.77$. □