Løsningsforslag øving 5
MA1202/6202 V2016

Kap 5.

Oppg 46 Kar. polynom

$$P_A(\lambda) = \begin{vmatrix} \lambda-1 & 0 & -9 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)^2$$

To egenverdier $\lambda_1 = 0$, $\lambda_2 = 1$

i) $\lambda_1 = 0$ gir

$$\begin{bmatrix} -1 & 0 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

dvs $x_1 = x_3 = 0$, x_2 kan velges fritt

Så egenrommet

$$V_0 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Så geometrisk multiplisitet $= \dim V_0 = 1$
algebraisk multiplisitet $= 1$.

ii) $\lambda = 1$ gir

$$\begin{bmatrix} 0 & 0 & -9 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

dvs. $x_2 = x_3 = 0$, x_1 kan velges fritt.

Så egenrommet

$$V_1 = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$$

Geometrisk multiplisitet $= \dim V_1 = 1$

Algebraisk multiplisitet $= 2$

<u>Konklusjon:</u> A kan ikke diagonaliseres.

<u>Oppg 53a</u> Kar. polynom.

$$P_A(\lambda) = \begin{vmatrix}\lambda-1 & 2 & -8\\ 0 & \lambda+1 & 0\\ 0 & 0 & \lambda+1\end{vmatrix} = (\lambda-1)(\lambda+1)^2$$

i) $\lambda = -1$ gir systemet

$$\begin{bmatrix}-2 & 2 & -8\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$x_2 = s$

$x_3 = t$

$x_1 = x_2 - 4x_3 = s - 4t$

$$\vec{x} = \begin{bmatrix}s-4t\\ s\\ t\end{bmatrix} = s\begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}-4\\0\\1\end{bmatrix}$$

Så $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}\right\}$ blir basis for V_{-1}, $\dim = 2$

ii) $\lambda = 1$ gir

$$\begin{bmatrix} 0 & 2 & -8 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

dvs $x_2 = x_3 = 0$
x_1 velges fritt.

$$V_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \quad \dim = 1.$$

Setter derfor

$$P = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

P^{-1} finner vi å se på systemet

$$\left[\begin{array}{ccc|ccc} 1 & -4 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 4 \end{array}\right]$$

dvs $P^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{bmatrix}$

Da blir $A = PDP^{-1}$ med

$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Så $A^{1000} = PD^{1000}P^{-1} = PIP^{-1} = I$

(strengt tatt behøvde vi ikke regne ut P^{-1}) Forøvrig $A^{2n+1} = A$, $A^{2n} = I$.

c) Så $A^{2301} = A$.

Oppg 77

$$p_A(\lambda) = \begin{vmatrix} \lambda-4 & 2 \\ -4 & \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 8$$

Kar. likning: $\lambda^2 - 4\lambda + 8 = 0$

$$\lambda = \frac{4 \pm \sqrt{16-32}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

i) $\lambda_1 = 2 + 2i$ gir

$$\begin{bmatrix} -2+2i & 2 \\ -4 & 2+2i \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2}(1+i) \\ 0 & 0 \end{bmatrix}$$

En egenvektor blir $\vec{v}_1 = \begin{pmatrix} \frac{1}{2}(1+i) \\ 1 \end{pmatrix}$

ii) $\lambda_2 = 2 - 2i$ gir

$$\begin{bmatrix} -2-2i & 2 \\ -4 & 2-2i \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2}(-1+i) \\ 0 & 0 \end{bmatrix}$$

Egenvektor $\vec{v}_2 = \begin{bmatrix} \frac{1}{2}(1-i) \\ 1 \end{bmatrix}$

iii) Så $\{\vec{v}_1, \vec{v}_2\}$ blir basis av egenvektorer.

Oppg 92 I $\mathbb{C}^n$ har vi

$$\|k\vec{v}\|^2 = k\vec{v}\cdot k\vec{v} = k\bar{k}\ \vec{v}\cdot\vec{v}$$

$$= |k|^2\|\vec{v}\|^2$$

Så $\|k\vec{v}\| = |k|\,\|\vec{v}\|$

Oppg 104 Sett $y_1 = y$ og $y_2 = y'$

Da blir $y_2' = y''$ som innsatt

i $y'' - 7y' + 6y = 0$ gir

$$y_1' = y' = y_2$$

$$y_2' = y'' = 7y' - 6y = -6y_1 + 7y_2$$

dvs $\vec{y}\,' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 7 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(e

Med $A = \begin{bmatrix} 0 & 1 \\ -6 & 7 \end{bmatrix}$ får vi

$$P_A(\lambda) = \begin{vmatrix} \lambda & -1 \\ 6 & \lambda-7 \end{vmatrix} = \lambda^2 - 7\lambda + 6$$

$$P_A(\lambda) = 0 \iff \lambda = \frac{7 \pm \sqrt{49-24}}{2} = 6, 1$$

i) $\lambda_1 = 1$ gir

$$\begin{bmatrix} 1 & -1 \\ 6 & -6 \end{bmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{egenvektor } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ii) $\lambda_2 = 6$ gir

$$\begin{bmatrix} 6 & -1 \\ 6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{6} \\ 0 & 0 \end{bmatrix} \quad \text{egenvektor } \begin{pmatrix} \frac{1}{6} \\ 1 \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & \frac{1}{6} \\ 1 & 1 \end{bmatrix} \quad (\text{trenger ikke } P^{-1})$$

Så med $D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$ er $A = PDP^{-1}$

Vi skal løse $(*)$ $\vec{y}\,' = A\vec{y}$.

~~Sett~~ $\vec{u} = P^{-1}\vec{y}$. Da blir $(*)$:

$$(**) \quad \vec{u}\,' = P^{-1}\vec{y}\,' = P^{-1}A\vec{y} = P^{-1}AP\vec{u} = D\vec{u}$$

(**) har løsning

$$\vec{u}(t) = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{6t} \end{bmatrix}$$

så

$$\vec{y}(t) = \begin{bmatrix} 1 & \frac{1}{6} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{t} \\ c_2 e^{6t} \end{bmatrix} = \begin{bmatrix} c_1 e^{t} + \frac{c_2}{6} e^{6t} \\ c_1 e^{t} + c_2 e^{6t} \end{bmatrix}$$

Så

$$y(t) = y_1(t) = \underline{c_1 e^{t} + c_2' e^{6t}}$$

$$\left(c_2' = \tfrac{1}{6} c_2 \right)$$

Kontroll: Stemmer med at $y_2 = y_1'$ $\boxtimes$

Eks V 2007. Opg 4

a) $A = \begin{bmatrix} -1 & 0 & 0 \\ -5 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 5 & \lambda+2 & -3 \\ 5 & 1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda+1)(\lambda+2)(\lambda-2) + 3(\lambda+1)$$

$$= (\lambda+1)(\lambda^2-4+3) = (\lambda+1)(\lambda^2-1)$$

$$= (\lambda+1)(\lambda+1)(\lambda-1)$$

Egenverdier : $\underline{\lambda_1 = -1 \;\&\; \lambda_2 = 1}$

$$\lambda_1 I - A = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 1 & -3 \\ 5 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 5 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = 3x_3 - 5x_1$$

Egenrommet tilhørende $\lambda_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

$$\lambda_2 I - A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & -3 \\ 5 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 = x_3 \quad x_1 = 0$$

Egenrommet tilhørende $\lambda_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

b) $P = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ diagonaliserer A, &

$$P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$$A^{97} = (PDP^{-1})^{97} = PD^{97}P^{-1} = PDP^{-1} = \underline{A}$$

$$A^{136} = (PDP^{-1})^{136} = PD^{136}P^{-1} = PIP^{-1} = \underline{I}$$

c)

$$\begin{bmatrix} -1 & 0 & 0 \\ -5 & -2 & 3 \\ -5 & -1 & 2 \end{bmatrix} \begin{bmatrix} g(x) \\ f(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} g'(x) \\ f'(x) \\ h'(x) \end{bmatrix}$$

$$A\underline{x} = \underline{x}'$$

$$PDP^{-1}\underline{x} = \underline{x}'$$

$$DP^{-1}\underline{x} = P^{-1}\underline{x}'$$

La $\underline{y} = P^{-1}\underline{x}$, $\underline{y} = \begin{bmatrix} a(x) \\ b(x) \\ c(x) \end{bmatrix}$

$$D\underline{y} = \underline{y}'$$

$$-a(x) = a'(x) \quad \Rightarrow \quad a(x) = re^{-x}$$
$$-b(x) = b'(x) \quad \Rightarrow \quad b(x) = se^{-x}$$
$$c(x) = c'(x) \quad \Rightarrow \quad c(x) = te^{x}$$

$$\underline{x} = P\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} re^{-x} \\ se^{-x} \\ te^{x} \end{bmatrix}$$

$$= \begin{bmatrix} re^{-x} \\ -5re^{-x} + 3se^{-x} + te^{x} \\ se^{-x} + te^{x} \end{bmatrix}$$

$$-1 = g(0) = r$$
$$1 = f(0) = -5r + 3s + t$$
$$2 = h(0) = s + t$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ -5 & 3 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 3 & 1 & -4 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -10 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right] \Rightarrow \begin{array}{l} g(x) = -e^{-x} \\ f(x) = -4e^{-x} + 5e^{x} \\ h(x) = -3e^{-x} + 5e^{x} \end{array}$$

Eksamen 2015. aug. Oppg 4

a) $$Y' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & -12 \\ 0 & 1 & -3 \end{bmatrix} Y$$

Så $$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & -12 \\ 0 & 1 & -3 \end{bmatrix}$$

b)

$$P_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda-4 & 12 \\ 0 & -1 & \lambda+3 \end{vmatrix}$$

$$= \lambda(\lambda+1)(\lambda-1)$$

OBS ikke lurt å multiplisere ut dette.

Så egenverdiene blir $\lambda_1 = 0$ $\lambda_2 = 1$ $\lambda_3 = -1$

i) $\lambda_1 = 0$, får matrisen

$$0I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 12 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Likningssystemet har løsning $x_1 = 0$ $x_2 - 3x_3 = 0$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ blir egenvektor

ii) $\lambda_2 = 1$ gir matrisen

$$I - A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 12 \\ 0 & -1 & 4 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

Likningssystemet har løsning $x_1 = 0$ $x_2 - 4x_3 = 0$

så $\vec{v}_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$ blir egenvektor

iii) $\lambda_3 = -1$ gir matrisen

$$-I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 12 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likningssystemet har løsning $x_2 = x_3 = 0$

der $\vec{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ er egenvektor.

iv) Derfor vil

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{gi} \quad P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

c) $Y' = AY = PDP^{-1}Y$ dvs

$$P^{-1}Y' = DP^{-1}Y.$$

sett $Z = P^{-1}Y$, så $Z' = P^{-1}Y'$

dvs $Z' = DZ$ i.e.

$$\begin{pmatrix} z_1'(x) \\ z_2'(x) \\ z_3'(x) \end{pmatrix} = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} z_1(x) \\ z_2(x) \\ z_3(x) \end{pmatrix}$$

dvs
$$z_1(x) = C_1$$
$$z_2(x) = C_2 e^{x} \quad \text{dvs } Z(x) = \begin{pmatrix} 1 & & \\ & e^{x} & \\ & & e^{-x} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$
$$z_3(x) = C_3 e^{-x}$$

Dette gir
$$Y(x) = PZ(x) = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 e^{x} \\ C_3 e^{-x} \end{pmatrix} = \begin{pmatrix} C_3 e^{-x} \\ 3C_1 + 4C_2 e^{x} \\ C_1 + C_2 e^{x} \end{pmatrix}$$

Så generell løsning blir:

$$y_1(x) = C_3 e^{-x}$$
$$y_2(x) = 3C_1 + 4C_2 e^{x}$$
$$y_3(x) = C_1 + C_2 e^{x}.$$

d) $y_1(0)=1 \quad y_2(0)=1 \quad y_3(0)=0$

gir

$$c_3 = 1$$

$$3c_1 + 4c_2 = 1$$

$$c_1 + c_2 = 0$$

Dette svarer til systemet

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 4 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

dvs $c_1 = -1 \quad c_2 = c_3 = 1$ dvs

$$Y(x) = \begin{bmatrix} e^{-x} \\ -3 + 4e^{x} \\ -1 + e^{x} \end{bmatrix}$$

Kontroller!