

Løsningsforslag Pringst
 Ma 1202/6202 V2016

Kap 6

Oppg 67d

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0 \right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + 0 = 0$$

Så vektorene er ortogonale.

68d Lengden av vektorene blir

$$\left(\frac{1}{6} + \frac{1}{6} + \frac{4}{6} \right)^{\frac{1}{2}} = 1 \quad \text{og} \quad \left(\frac{1}{2} + \frac{1}{2} + 0 \right)^{\frac{1}{2}} = \sqrt{2}$$

Så vi har en orthonormal mengde.

Oppg 69b $\langle P_1, P_2 \rangle = 0$ $\langle P_1, P_3 \rangle = 0$

$$\langle P_2, P_3 \rangle = \frac{1}{\sqrt{2}} \quad \|P_1\| = \|P_2\| = \|P_3\| = 1.$$

Så $\{P_1, P_2, P_3\}$ er ikke orthonormal.

Oppg 81b $\vec{x} \cdot \vec{v}_1 = 1$ $\vec{x} \cdot \vec{v}_2 = 2$ $\vec{x} \cdot \vec{v}_3 = 0$.

Så

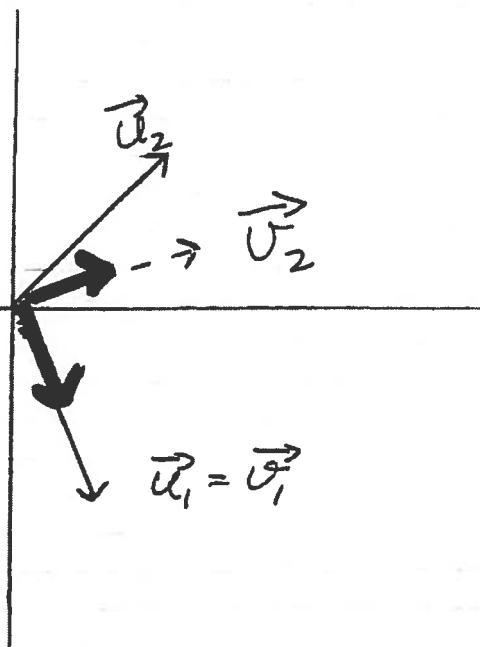
$$\text{proj}_{\mathcal{W}}(\vec{x}) = 1 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \frac{1}{2}(3, 3, -1, -1)$$

Oppg 84 $\vec{v}_1 = \vec{u}_1 = (1, -3)$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 = (2, 2) - \frac{-4}{10} (1, -3) = \frac{4}{5} (3, 1)$$

Orthonormal basis: $\frac{1}{\sqrt{10}}(1, -3)$ og $\frac{1}{\sqrt{10}}(3, 1)$.

Oppg 84 (Forbry)



→ ortonormale basis

$$\text{Oppg 46} \quad P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = x^2$$

$$\text{Så } \underline{q_0(x) = 1}$$

$$\langle P_1, q_0 \rangle = \int_0^1 x \cdot 1 dx = \frac{1}{2} \quad \langle q_0, q_0 \rangle = \int_0^1 1 \cdot 1 dx = 1$$

$$\text{Så } q_1 = P_1 - \frac{\langle P_1, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 = P_1 - \frac{1}{2} q_0$$

$$\text{Dvs } \underline{q_1(x) = x - \frac{1}{2}}$$

$$\langle P_2, q_0 \rangle = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3} \quad \langle P_2, q_1 \rangle = \int_0^1 x^2 (x - \frac{1}{2}) dx = \frac{1}{12}$$

$$\langle q_1, q_1 \rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$$

Så

$$q_2 = P_2 - \frac{\langle P_2, q_0 \rangle}{\langle q_0, q_0 \rangle} q_0 - \frac{\langle P_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 \\ = P_2 - \frac{1}{3} q_0 - q_1$$

$$\underline{q_2(x) = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}}$$

$$\langle q_2, q_2 \rangle = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \frac{1}{180}$$

Orthonormal basis blir da

$$r_0(x) = 1, \quad r_1(x) = \frac{1}{\sqrt{3}} \cdot (x - \frac{1}{2})$$

$$r_2(x) = \frac{1}{\sqrt{5}} \cdot (x^2 - x + \frac{1}{6})$$

Uppg 100a Vi ska lösa $A^T A \vec{x} = A^T b$
som ger

$$\begin{bmatrix} 6 & 3 & 7 \\ 3 & 5 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{9}{7} \end{bmatrix} \text{ dvs } \vec{x} = \frac{1}{2} \begin{bmatrix} 11 \\ 27 \end{bmatrix}$$

102a

$$\vec{e} = \vec{b} - A\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \cdot \frac{1}{21} \begin{bmatrix} 11 \\ 27 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0 \quad \begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0$$

så $\vec{e} \perp \text{Col}(A)$.

Uppg 104c Mark $A\vec{x} = \vec{b}$ har ikke lösning.

$$A^T A \vec{x} = A^T \vec{b} \text{ ger}$$

$$\begin{bmatrix} 5 & -1 & 4 & -7 \\ -1 & 11 & 10 & 14 \\ 4 & 10 & 14 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -\frac{7}{6} \\ 0 & 1 & 1 & \frac{7}{6} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{dvs } \vec{x} = \frac{7}{6} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \text{ Där } A\vec{x} = \frac{7}{6} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{e} = \vec{b} - A\vec{x} = \begin{bmatrix} 7 \\ 0 \\ -7 \end{bmatrix} - \frac{7}{6} \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix} = \frac{7}{6} \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$$

$$\text{Fel} = \|\vec{e}\| = \frac{7}{2} \sqrt{6}.$$

Opgang 105 a

Må først finne en ortogonal basis for
 $W = \text{span}\{\vec{v}_1, \vec{v}_2\}$

Brukbar Gram-Schmidt:

$$\vec{w}_1 = \vec{v}_1 = (2, 1, 0)$$

$$\vec{w}_2' = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1$$

$$= (-1, 1, 0) - \frac{-1}{5}(2, 1, 0) = \frac{3}{5}(-1, 2, 0)$$

Kan like gjevne brukke $\vec{w}_2 = (-1, 2, 0)$.

Da blir

$$\text{proj}_W(\vec{u}) = \frac{\langle \vec{u}, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 + \frac{\langle \vec{u}, \vec{w}_2 \rangle}{\langle \vec{w}_2, \vec{w}_2 \rangle} \vec{w}_2$$

$$= \frac{4}{5}(2, 1, 0) + \frac{3}{5}(-1, 2, 0)$$

$$= (1, 2, 0)$$

Kunne også sees direkte, siden

$$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$$

Oppgave 3(a). Vi følger hintet og viser at $f(x)$ og $g(x)$ er ortogonale:

$$\begin{aligned}\langle f(x), g(x) \rangle &= f(0)g(0) + f(1)g(1) + f(-1)g(-1) \\ &= 0 \cdot 1 + 4 \cdot (-4) + 2 \cdot 8 \\ &= 0.\end{aligned}$$

Følgelig utgjør mengden $\{f(x), g(x)\}$ en ortogonal basis for underrommet W av P_2 . Da har vi et resultat som sier at $\text{proj}_W h(x)$ er gitt ved

$$\text{proj}_W h(x) = \frac{\langle f(x), h(x) \rangle}{\|f(x)\|^2} f(x) + \frac{\langle g(x), h(x) \rangle}{\|g(x)\|^2} g(x)$$

hvor $\|f(x)\|^2 = \langle f(x), f(x) \rangle$. For å regne ut alt dette trenger vi å vite verdiene til de tre funksjonene for $x = 0, 1, -1$:

$$\begin{array}{lll}f(0) = 0 & g(0) = 1 & h(0) = 1 \\ f(1) = 4 & g(1) = -4 & h(1) = 3 \\ f(-1) = 2 & g(-1) = 8 & h(-1) = 1\end{array}$$

De fire indreproduktene vi trenger blir da

$$\begin{array}{lll}\langle f(x), h(x) \rangle &= 14 \\ \langle f(x), f(x) \rangle &= 20 \\ \langle g(x), h(x) \rangle &= -3 \\ \langle g(x), g(x) \rangle &= 81\end{array}$$

Projeksjonen av $h(x)$ ned på underrommet W er derfor gitt ved

$$\begin{aligned}\text{proj}_W h(x) &= \frac{14}{20} f(x) + \frac{(-3)}{81} g(x) \\ &= \frac{7}{10}(x + 3x^2) - \frac{1}{27}(1 - 6x + x^2) \\ &= -\frac{1}{27} + \frac{83}{90}x + \frac{557}{270}x^2.\end{aligned}$$

Oppgave 3(b). Det er fire krav til et indreprodukt, og en del av det siste kravet er at ekvivalensen

$$\langle p(x), p(x) \rangle = 0 \Leftrightarrow p(x) = 0$$

skal gjelde for alle $p(x) \in P_2$. Ta nå polynomet

$$p(x) = (x - 3)(x - 4) = 12 - 7x + x^2$$

i P_2 . Da har vi:

$$\begin{array}{ll}p(x) &\neq 0 \\ \langle p(x), p(x) \rangle &= 0\end{array}$$

siden $p(3) = 0 = p(4)$. Derfor er dette ikke et indreprodukt på P_2 .

Problem 2 Consider the following vectors in \mathbb{R}^4 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix},$$

and let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

- a) Are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent?
Find a basis for V .

Solution. We consider the matrix

$$A = [\mathbf{v}_1 | \mathbf{v}_2 | \mathbf{v}_3 | \mathbf{v}_4] = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 2 & 1 & -2 & 1 \\ 0 & -1 & 4 & 1 \end{bmatrix},$$

whose columns are the given vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and \mathbf{v}_4 .

Since

$$\text{column space of } A = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = V,$$

we will find a basis for V by finding a basis for the column space of A .

We use Gaussian elimination to find the reduced row echelon form (RREF) of A :

$$\begin{array}{c} \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 2 & 1 & -2 & 1 \\ 0 & -1 & 4 & 1 \end{array} \right] \xrightarrow{-2R_1+R_3} \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 4 & 1 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \\ \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & -1 & 4 & 1 \end{array} \right] \xrightarrow{R_2+R_4} \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -6 & -4 \end{array} \right] \xrightarrow{-\frac{1}{3}R_3} \\ \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & -6 & -4 \end{array} \right] \xrightarrow{6R_3+R_4} \left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{array}$$

This is the reduced echelon form (REF).

We perform one more step to obtain the RREF:

$$\left[\begin{array}{cccc} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-4R_3+R_1 \\ 10R_3+R_2}} \left[\begin{array}{cccc} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore, the RREF of A is

$$R = \left[\begin{array}{cccc} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{array} \right] = [\mathbf{c}_1 | \mathbf{c}_2 | \mathbf{c}_3 | \mathbf{c}_4].$$

Note the following relation between the columns of R :

$$\mathbf{c}_4 = \frac{1}{3}\mathbf{c}_1 + \frac{5}{3}\mathbf{c}_2 + \frac{2}{3}\mathbf{c}_3.$$

The same relation will hold between the columns of the original matrix A , in other words:

$$\mathbf{v}_4 = \frac{1}{3}\mathbf{v}_1 + \frac{5}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3,$$

showing that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are *not* linearly independent.

There are, of course, other arguments for why these vectors are not linearly independent, for instance because the determinant of A is 0 (compute it!).

Now regarding the basis for $V = \text{span} \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \}$.

The columns with leading 1's in R are the 1st, 2nd and 3rd, so the *corresponding* 1st, 2nd and 3rd columns of A form a basis for the column space of A .

In conclusion: a basis for V is $\mathcal{B} = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$. □

b) Find an orthogonal basis for V .

Solution. We apply the Gram-Schmidt process to the basis \mathcal{B} determined above, to obtain an *orthogonal* basis $\mathcal{B}' = \{ \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3 \}$.

Step 1:

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1.$$

We compute separately

$$\langle \mathbf{v}_2, \mathbf{w}_1 \rangle = \langle \mathbf{v}_2, \mathbf{v}_1 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\rangle = 2.$$

$$\|\mathbf{w}_1\|^2 = \|\mathbf{v}_1\|^2 = 1^2 + 2^2 = 5.$$

Then

$$\mathbf{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{5} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix}$$

Step 2:

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\|\mathbf{w}_2\|^2} \mathbf{w}_2.$$

We compute separately

$$\langle \mathbf{v}_3, \mathbf{w}_1 \rangle = \left\langle \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} \right\rangle = 4 - 4 = 0.$$

$$\langle \mathbf{v}_3, \mathbf{w}_2 \rangle = \left\langle \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix} \right\rangle = -8/5 - 2/5 - 4 = -6.$$

$$\|\mathbf{w}_2\|^2 = 4/25 + 1/25 + 1 = 6/5.$$

Then

$$\mathbf{w}_3 = \begin{bmatrix} 4 \\ -3 \\ -2 \\ 4 \end{bmatrix} - 0 - \frac{-6}{6/5} \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix}.$$

Therefore, an orthogonal basis for V is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2/5 \\ 0 \\ 1/5 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix} \right\}.$$

□

- c) Does there exist a non-zero vector \mathbf{u} in \mathbb{R}^4 which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

Solution. In question 1b) we have obtained an orthogonal basis $\mathcal{B}' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the subspace V of \mathbb{R}^4 . Since $\dim(V) = 3$ and $\dim(\mathbb{R}^4) = 4$, \mathcal{B}' can be *enlarged* to an orthogonal basis for \mathbb{R}^4 , by adding one extra vector \mathbf{u} .

As an element of a basis, \mathbf{u} must be non-zero.

As an element of an orthogonal basis in \mathbb{R}^4 , \mathbf{u} must be orthogonal to all other elements $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of that basis. Since $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for the subspace V , \mathbf{u} is in fact orthogonal to *all* vectors in this subspace, and in particular it must be orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, since they are elements of V .

Therefore, to answer to the question is *yes*, such a vector exists. □