The following is a special case of Dirichlet's Theorem.

Proposition 1. *(Proposition 3.38 in Erickson/Vazzana).*
There are infinitely many primes of the form $4k+1$*.*

Remark: The case $4k+3$ is Proposition 2.38.

Proof. Let $N \geq 2$. None of the numbers $2, 3, 4, \ldots, N$ divide $P = (2 \cdot 3 \cdot 4 \cdot \ldots \cdot N)^2 + 1$. Then for any prime factor p of P we have $p > N$. Now

$$p | (N!)^2 + 1 \Leftrightarrow (N!)^2 \equiv -1 \mod p$$

so $x = N!$ is a solution to the quadratic congruence

$$x^2 \equiv -1 \mod p.$$

But by Theorem 3.37 this congruence only has solutions if $p = 2$ or $p = 4k+1$. Since $p > N \geq 2$, we have $p = 4k+1$. So for any N there exists a prime p with $p > N$ and $p = 4k+1$. Hence there are infinitely many primes of the form $4k+1$. □

Remark: The first steps are

$$2^2 + 1 = 5$$

$$(3!)^2 + 1 = 37$$

$$(4!)^2 + 1 = 577$$

$$(5!)^2 + 1 = 14401$$

$$(6!)^2 + 1 = 518401 = 13 \cdot 39877.$$