MA2501 Numerical methods

Assignment 10

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Exercise 1

Given the equation

$$x'' + x = 0, \qquad x(0) = 1, \qquad x'(0) = 0, \qquad t = [0, 10] \tag{1}$$

The exact solution of this equation is given by $x(t) = \cos(t)$, thus, the solution in the phase plane (x, x') is a circle. This is a property we also want from the the numerical solution.

a) Write the equation as a system of 1.order differential equations (see chap. 11.2 i C&K).

b) Solve the system by the forward Euler method. Plot the solution in the phase plane. What happens? Use different stepsizes and see how the numerical solution changes.

c) Repeat point b), but now with backward Euler ($x_{i+1} = x_i + hf(t_{i+1}, x_{i+1})$). What happens now?

d) Alternate between the two methods (one step with forward Euler, one with the backward method). Is the solution improved?

e) Let $x'(t) = y(t)$. Prove that the exact solution of (1) can be written as

$$\begin{bmatrix} x(t_i + h) \\ y(t_i + h) \end{bmatrix} = \begin{bmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{bmatrix} \begin{bmatrix} x(t_i) \\ y(t_i) \end{bmatrix}$$

Use this to prove that $\|[x(t_i + h), y(t_i + h)]\|_2 = \|[x(t_i), y(t_i)]\|_2$ independent of the stepsize h.

f) One step with the forward Euler method applied to (1) can be written as

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = r \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

What is r and θ? Use this to explain the behaviour in b).

g) Repeat f) for backward Euler.

h) Can you now explain why the result is much better when alternating between the two methods?

Exercise 2

Given a system of ordinary differential equations

$$\mathbf{x}' = \mathbf{F}(t, \mathbf{x}), \qquad \mathbf{x}(t_0) = \mathbf{x}_0.$$

One step of an s-stage Runge-Kutta method for this equation is given by

$$\mathbf{K}_i = h\mathbf{F}(t_0 + hc_i, \mathbf{x}_0 + \sum_{j=1}^{s} a_{ij}\mathbf{K}_j), \qquad i = 1, \cdots, s$$

$$\mathbf{x}_1 = \mathbf{x}_0 + h\sum_{i=1}^{s} b_i\mathbf{K_i}$$

Each methods is defined by the coefficients c_i, a_{ij} and b_i. These are usually written in a Butcher-tableau

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\ c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & \vdots & \vdots & & \vdots \\ c_s & a_{s1} & a_{s2} & \cdots & a_{ss} \\ \hline & b_1 & b_2 & \cdots & b_s. \end{array}$$

The method is *explicit* if $a_{ij} = 0$ for $j \geq i$, otherwise implicit.

The tableau for forward and backward Euler respectively is

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}, \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}.$$

a) The alternating methods described in Exercise 1 d) can be written as two different Runge-Kutta methods, depending of whether we start with forward or backward Euler. Show how, and find the Butcher-tableau of the two methods.

b) What is the order of the two methods (use the notes about Runge-Kutta methods).

Exercise 3

Show that the order of an s-stage explicit Runge-Kutta method can never exceed s.
Hint: Use as a test case $x' = x,\ x(0) = 1$.