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EXAM IN NUMERICAL METHODS (MA2501)

Friday the 2nd of June 2006
Time: 09:00 – 13:00 Examination results due on Friday the 23rd of June 2006
Permitted technical aids:

- W. Cheney & D. Kincaid, *Numerical Mathematics and Computing*, 4. eller 5. edition.
- Pocket calculator.

General information:

- All subproblems count equally towards the examination results.
- All answers must be justified.
- Any answer must be accompanied by sufficient calculations to allow assessement of which methods and partial results are used.
- If two functions $f_1(x)$ and $f_2(x)$ are sufficiently differentiable, then

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(f_1(x)\,f_2(x)\big) = \sum_{k=0}^{n} \binom{n}{k} f_1^{(k)}(x)\,f_2^{(n-k)}(x) \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

- The attachment at the end of the problem set is required in the answer to Problem 4.
- Good luck!

Problem 1 The function

$$f(x) = \frac{\mathrm{e}^{-x}}{1+x}$$

is defined for all $x \geq 0$.

a) Find the minimum degree polynomial $p(x)$ which interpolates $f(x)$ at the points $x_0 = 0$, $x_1 = 2$, $x_2 = 6$, and $x_3 = 8$. Compute $f(1/2) - p(1/2)$.

b) Determine a guaranteed upper bound on the error $|f(x) - p(x)|$ when $0 \leq x \leq 8$. Is this bound reasonable?

Problem 2 The function

$$f(x) = \frac{\mathrm{e}^{-x}}{1+x}$$

is defined for all $x \geq 0$.

a) Find an approximation to $\int_0^8 f(x)\,\mathrm{d}x$ using Simpson's method with 8 sub-intervals.

b) We wish to compute $\int_0^\infty f(x)\,\mathrm{d}x$. One way of doing this is to compute, by means of Simpson's method using step size h, the integral

$$\int_0^B f(x)\,\mathrm{d}x$$

with $B > 0$ fixed and finite.

Show that the error committed in this approach is bounded from above by

$$E_1(h, B) + E_2(B) = \tfrac{13}{36} B h^4 + \mathrm{e}^{-B}.$$

c) Determine the value of B and the least number of sub-intervals, n, which guarantee that the error of the method in **b**) is less than $\frac{1}{2} \cdot 10^{-3}$ (i.e. the number of correct desimals in the result is at least three).

Problem 3 The function

$$R(z) = \frac{1 + z/2}{1 - z/2} \tag{1}$$

is defined for all $-2 < z < 2$.

a) Prove that $\mathrm{e}^z - R(z) = -\frac{z^3}{12} + \mathcal{O}(z^4)$.

b) Find a unit diagonal, lower triangular matrix L and an upper triangular matrix U such that $LU = A$ when

$$A = \begin{bmatrix} 1.10 & -0.05 & 0.00 \\ -0.05 & 1.10 & -0.05 \\ 0.00 & -0.05 & 1.10 \end{bmatrix}.$$

Hint: Pivoting is not needed.

c) The *matrix exponential* e^{hX} in which X is a square matrix and $h \in \mathbb{R}$ is a scalar is defined by the series expansion

$$\mathrm{e}^{hX} = I + \sum_{n=1}^{\infty} \frac{(hX)^n}{n!}. \tag{2}$$

Here, I is the identity matrix of the same size as X. We remark that this series converges for all square matrices X. The matrix exponential is an example of a generalisation of a known, analytic function to matricial arguments. The function is central to certain types of methods for the resolution of ordinary differential equations and we are hence in need of good methods to approximate e^{hX}.

To this end several algorithms have been proposed, among which we find terminated series expansions based on (2). In this problem, however, we will use *rational approximations* to e^{hX}. Specifically, we will evaluate $R(hX)$ with $R(z)$ defined in (1).

State the linear system of equations to which $R(hX)$ is a solution when

$$h = 0.1, \quad X = \begin{bmatrix} -2 & 1 & \\ 1 & -2 & 1 \\ & 1 & -2 \end{bmatrix}.$$

Hint: The constant 1 must be replaced by the identity matrix when evaluating $R(z)$ at matricial arguments.

Compute ***the second column*** of $R(hX)$.

Problem 4 The partial differential equation and accompanying initial and boundary conditions

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{1}{1+u^2}, \quad (x,t) \in [0,1] \times [0,1] \\ u(0,t) &= 0, \quad u(1,t) = 0 \\ u(x,0) &= x \cdot (1-x), \quad 0 \le x \le 1\end{aligned}$$

is given. Let $x_i = ih$ for all $i = 0, 1, \ldots, N$ be grid points in the x direction at a distance of $h = 1/N$ apart. Furthermore, let $U_i(t)$ be an approximation to $u(x_i, t)$ for all $i = 0, 1, \ldots, N$ and all $t \in [0, 1]$.

a) Derive a system of *ordinary* differential equations for the functions $U_i(t)$ by, among other things, replacing all spatial derivatives (i.e. terms of the form $\frac{\partial u}{\partial x}$ and similar) of the partial differential equation by suitable difference approximations.

Use the dependent variable $\mathbf{y} = [U_1, U_2, \ldots, U_{N-1}]^\mathsf{T}$ in the resulting system.

b) Write a MATLAB function named `ode_rhs` which, toghether with MATLAB's built-in ODE solver `ode15s`, can be used in the resolution of the system of ODEs derived in **a**). See the attachment for detailed information on `ode15s`.

If you were unsuccessful in the derivation of problem **a**), you will here consider the ODE system defined by

$$\begin{aligned} y_1' &= -\tfrac{N}{4} y_2^2 + 10^{-3} N^2 \big(-2y_1 + y_2 \big) \\ y_i' &= \tfrac{N}{4} (y_{i-1}^2 - y_{i+1}^2) + 10^{-3} N^2 \big(y_{i-1} - 2y_i + y_{i+1} \big), \quad i = 2, 3, \ldots, N-2 \\ y_{N-1}' &= \tfrac{N}{4} y_{N-2}^2 + 10^{-3} N^2 \big(y_{N-2} - 2y_{N-1} \big). \end{aligned}$$

Finally, give the MATLAB statements required to resolve the partial differential equation by means of MATLAB's built-in function `ode15s`. Use $N = 200$.

We note in conclusion that the general methodolgy illustrated in this problem is called semi-discretisation.

Enjoy your summer!

The MATLAB ODE solver `ode15s`

`[T, Y] = ODE15S(ODEFUN, TSPAN, Y0)`

If `TSPAN = [T0 TFINAL]`, this call integrates the system of differential equations

$$\begin{cases} \mathbf{y}' = \mathbf{f}(t, \mathbf{y}) \\ \mathbf{y}(t_0) = \mathbf{y}_0 \end{cases}$$

from time `T0` to `TFINAL` using initial conditions `Y0`.

The function `ODEFUN(T,Y)` must return a *column* vector corresponding to $\mathbf{f}(t, \mathbf{y})$. Each row in the solution array `Y` corresponds to a time returned in the column vector `T`.

To obtain solutions at specific times `T0,T1,...,TFINAL` (all increasing or all decreasing), use `TSPAN = [T0 T1 ... TFINAL]`.