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Bounding $|\zeta(\frac{1}{2}+it)|$ on the Riemann hypothesis

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Abstract

In 1924 Littlewood showed that, assuming the Riemann hypothesis, for large t, there is a constant C such that $|\zeta(1/2+it)| \ll \exp(C\log t/\log\log t)$. In this note we show how the problem of bounding $|\zeta(1/2+it)|$ may be framed in terms of minorizing the function $\log((4+x^2)/x^2)$ by functions whose Fourier transforms are supported in a given interval, and drawing upon recent work of Carneiro and Vaaler we find the optimal such minorant. Thus we establish that any $C > (\log 2)/2$ is permissible in Littlewood's result.

1. *Introduction*

In 1924 Littlewood [10] proved that the Riemann hypothesis (RH) implies a strong form of the Lindelöf hypothesis; namely, on RH, for large t, there is a constant C such that

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(C\frac{\log t}{\log\log t}\right). \tag{1}$$

In the intervening years no improvement has been made over (1), except in reducing the permissible value of C, see [11, 12]. In [12] Soundararajan showed that (1) holds for any $C > (1+\lambda_0)/4 = 0.372\ldots$ where $\lambda_0 = 0.4912\ldots$ is the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0 + \lambda_0^2/2$. In [4] Chandee has provided an explicit version of this bound for general L-functions.

A similar situation exists for $S(t) = (1/\pi)\arg\zeta(\frac{1}{2}+it)$, where the argument is defined by continuous variation along the line segments joining 2, $2+it$, and $\frac{1}{2}+it$, taking the argument of $\zeta(s)$ at 2 to be zero. On RH Littlewood showed that $S(t) \ll \log t/\log\log t$, and again this bound has not been improved except for the size of the implied constant. Recently Goldston and Gonek [7] gave an elegant argument leading to the bound $|S(t)| \leqslant (\frac{1}{2}+o(1))\log t/\log\log t$. Their method used the explicit formula together with certain optimal majorants and minorants of characteristic functions of intervals that were constructed by Selberg. The Goldston–Gonek result may reasonably be thought of as having attained the limit of existing methods of bounding $S(t)$, although it seems likely that the true maximal size of $S(t)$ is even smaller, perhaps $\ll \sqrt{\log t\log\log t}$ (see [6]).

In [12] Soundararajan asked for a corresponding treatment for $|\zeta(\frac{1}{2}+it)|$ which would represent the limit of existing methods for bounding $|\zeta(\frac{1}{2}+it)|$ on RH. In this note we present such an approach. Using Hadamard's factorization formula and the explicit formula, we show how the problem of bounding $|\zeta(\frac{1}{2}+it)|$ may be framed in terms of minorizing the function $\log((4+x^2)/x^2)$ by functions whose Fourier transforms are supported in a given interval, and drawing upon recent work of Carneiro and Vaaler [3] we find the optimal such minorant.

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THEOREM 1.1. *Assume RH. For large real numbers t we have*

$$\left|\zeta\left(\frac{1}{2}+it\right)\right| \ll \exp\left(\frac{\log 2}{2}\frac{\log t}{\log\log t} + O\left(\frac{\log t \log\log\log t}{(\log\log t)^2}\right)\right).$$

As with $S(t)$, the true maximal size of $|\zeta(\frac{1}{2}+it)|$ may be much smaller, perhaps of size $\exp(\sqrt{(\frac{1}{2}+o(1))\log t\log\log t})$ as suggested by Farmer, Gonek, and Hughes [**6**]. On the other hand, it is known that there are arbitrarily large t such that $|\zeta(\frac{1}{2}+it)| \geqslant \exp((1+o(1))\sqrt{\log t/\log\log t})$; see [**13**].

2. *Proof of Theorem* 1.1

Let $\xi(s) = (1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ denote Riemann's ξ-function which is entire of order 1, satisfies the functional equation $\xi(s) = \xi(1-s)$, and whose zeros are the non-trivial zeros of $\zeta(s)$. Recall (see, for example, [**5**, Chapter 12]) Hadamard's factorization formula

$$\xi(s) = e^{A+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where ρ runs over the non-trivial zeros of $\zeta(s)$, and $B = -\sum_\rho \mathrm{Re}(1/\rho)$. (Note that $\mathrm{Re}(1/\rho)$ is positive and $\sum_\rho \mathrm{Re}(1/\rho)$ converges.) We apply this with $s = \frac{1}{2}+it$ and $s = -\frac{3}{2}+it$ and divide. The absolute convergence of the product allows us to divide term by term, and we find, writing (on RH) $\rho = \frac{1}{2}+i\gamma$,

$$\left|\frac{\xi(1/2+it)}{\xi(-3/2+it)}\right| = e^{2B}\prod_\rho\left|\frac{i(\gamma-t)}{2+i(\gamma-t)}\right|e^{\mathrm{Re}(2/\rho)} = \prod_\rho\left|\frac{(t-\gamma)^2}{4+(t-\gamma)^2}\right|^{1/2}.$$

Since $\xi(-\frac{3}{2}+it) = \xi(\frac{5}{2}-it)$, and $|\zeta(\frac{5}{2}-it)| \asymp 1$, we deduce using Stirling's formula that

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| = \log t + O(1) - \frac{1}{2}\sum_\gamma f(t-\gamma), \tag{2}$$

where we have set

$$f(x) = \log\frac{4+x^2}{x^2}. \tag{3}$$

The proof of Theorem 1.1 now proceeds by replacing $f(t-\gamma)$ by a carefully chosen function that minorizes it, and then invoking the explicit formula. The properties of the appropriate minorant function are detailed in the following proposition which we shall demonstrate in the next section.

PROPOSITION 2.1. *Let Δ denote a positive real number. There is an entire function g_Δ which satisfies the following properties.*

(i) *For all real x we have*

$$-C\frac{1}{1+x^2} \leqslant g_\Delta(x) \leqslant f(x),$$

for some positive constant C. For any complex number $x+iy$ we have

$$|g_\Delta(x+iy)| \ll \frac{\Delta^2}{1+\Delta|x+iy|}e^{2\pi\Delta|y|}.$$

(ii) *The Fourier transform of* g_Δ, *namely*

$$\hat{g}_\Delta(\xi) = \int_{-\infty}^{\infty} g_\Delta(x) e^{-2\pi i x\xi}\, dx,$$

is real-valued, equals zero for $|\xi| \geqslant \Delta$, *and satisfies* $|\hat{g}_\Delta(\xi)| \ll 1$.

(iii) *The* L^1 *distance between* g_Δ *and* f *equals*

$$\int_{-\infty}^{\infty} (f(x) - g_\Delta(x))\, dx = \frac{1}{\Delta}(2\log 2 - 2\log(1+e^{-4\pi\Delta})).$$

Returning to (2), we have for any positive Δ

$$\sum_\gamma f(t-\gamma) \geqslant \sum_\gamma g_\Delta(t-\gamma). \tag{4}$$

We now invoke the explicit formula connecting zeros and primes (see [**7**, Lemma 1] or [**9**, Theorem 5.12]).

LEMMA 2.2. *Let* $h(s)$ *be analytic in the strip* $|\mathrm{Im}\ s| \leqslant 1/2 + \epsilon$ *for some* $\epsilon > 0$, *and such that* $|h(s)| \ll (1+|s|)^{-(1+\delta)}$ *for some* $\delta > 0$ *when* $|\mathrm{Re}\ s| \to \infty$. *Let* $h(w)$ *be real-valued for real* w, *and set* $\hat{h}(x) = \int_{-\infty}^{\infty} h(w) e^{-2\pi i x w}\, dw$. *Then*

$$\sum_\rho h(\gamma) = h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right) - \frac{1}{2\pi}\hat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u) \mathit{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) du$$
$$- \frac{1}{2\pi}\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(\frac{-\log n}{2\pi}\right)\right).$$

We apply Lemma 2.2, taking $h(z) = g_\Delta(t-z)$ so that $\hat{h}(x) = \hat{g}_\Delta(-x)e^{-2\pi i x t}$. From (i) of Proposition 2.1 we find that $h(1/2i) + h(-1/2i) \ll \Delta^2 e^{\pi\Delta}/(1+\Delta t)$, and using (ii) of Proposition 2.1 that $\hat{h}(0) \ll 1$. Using Stirling's formula, parts (i) and (iii) of Proposition 2.1, and that $\int_{-\infty}^{\infty} f(x)dx = 4\pi$ we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} h(u) \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) du = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_\Delta(u)(\log t + O(\log(2+|u|)))\, du$$
$$= 2\log t - \frac{\log t}{\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + O(1).$$

Using these remarks to evaluate the right-hand side of (4), and inserting that bound in (2) we conclude that

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant \frac{\log t}{2\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right) + \frac{1}{2\pi}\mathrm{Re}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{1/2+it}}\hat{g}_\Delta\left(\frac{\log n}{2\pi}\right)$$
$$+ O\left(\frac{\Delta^2 e^{\pi\Delta}}{1+\Delta t} + 1\right). \tag{5}$$

Since

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left|\hat{g}_\Delta\left(\frac{\log n}{2\pi}\right)\right| \ll e^{\pi\Delta},$$

taking $\pi\Delta = \log\log t - 3\log\log\log t$ in (5) we obtain our theorem.

3. *Proof of Proposition* 2.1: *the work of Carneiro and Vaaler*

Given a function from $\mathbb{R}$ to $\mathbb{R}$ Carneiro and Vaaler consider the problem of finding optimal majorants and minorants for this function, with the additional property that the majorants and minorants are restrictions to the real axis of complex analytic functions of exponential type at most 2π. The majorants and minorants are to be optimal in the sense of minimizing the L^1 distance from the given function. This problem has a long history, going back to work of Beurling for the signum function which was rediscovered and used by Selberg to study the case of indicator functions of intervals (see [**3**, **8**, **14**]). Carneiro and Vaaler solve the optimization problem for a wide class of functions including our function $f(x)$.

Let μ be a (non-negative) measure defined on the Borel subsets of $\mathbb{R}_+$ such that

$$0 < \int_0^\infty \frac{\lambda}{\lambda^2+1}\, d\mu(\lambda) < \infty. \tag{6}$$

Let

$$f_\mu(x) = \int_0^\infty (e^{-\lambda|x|} - e^{-\lambda})\, d\mu(\lambda),$$

and define

$$G_\mu(z) = \lim_{N\to\infty} \Big(\frac{\cos \pi z}{\pi}\Big)^2 \sum_{n=-N}^{N+1} \Big(\frac{f_\mu(n-1/2)}{(z-n+1/2)^2} + \frac{f'_\mu(n-1/2)}{(z-n+1/2)} \Big).$$

Theorem 1.1 of Carneiro and Vaaler then demonstrates that $G_\mu(z)$ converges uniformly on compact subsets of $\mathbb{C}$, defines an entire function of exponential type at most 2π, and that for real x we have $G_\mu(x) \leqslant f_\mu(x)$. Moreover they show that G_μ minimizes the L^1 distance from f_μ (in particular $f_\mu - G_\mu$ is integrable) among all minorants of f_μ with exponential type at most 2π.

Let Δ be a given positive real number, and consider the measure

$$d\mu_\Delta(\lambda) = \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\, d\lambda.$$

This measure satisfies (6), and moreover

$$\int_0^\infty (e^{-\lambda|x|} - e^{-\lambda}) \frac{2(1-\cos(2\Delta\lambda))}{\lambda}\, d\lambda = \log\Big(\frac{4\Delta^2+x^2}{x^2}\Big) - \log(4\Delta^2+1). \tag{7}$$

The identity (7) may be checked by noting that both sides equal zero for $x=1$, and that the derivatives of both sides agree (a little care is needed at $x=0$ where the result follows by continuity). Let us denote the right-hand side of (7) by $f_\Delta(x) = f(x/\Delta) - f(1/\Delta)$. Let

$$G_\Delta(z) = \Big(\frac{\cos \pi z}{\pi}\Big)^2 \sum_{n=-\infty}^{\infty} \Big(\frac{f_\Delta(n-1/2)}{(z-n+1/2)^2} + \frac{f'_\Delta(n-1/2)}{(z-n+1/2)} \Big)$$

denote the corresponding optimal function of Carneiro and Vaaler.

First we record an upper bound for $G_\Delta(z)$. By an application of the Poisson summation formula we see that

$$\Big(\frac{\cos \pi z}{\pi}\Big)^2 \sum_{n=-\infty}^{\infty} \frac{1}{(z-n+1/2)^2} = \sum_{n=-\infty}^{\infty} \Big(\frac{\sin(\pi(z-n+1/2))}{\pi(z-n+1/2)}\Big)^2 = 1,$$

so that

$$G_\Delta(z) + f\Big(\frac{1}{\Delta}\Big) = \sum_{n=-\infty}^{\infty} \Big(\frac{\sin(\pi(z-n+1/2))}{\pi(z-n+1/2)}\Big)^2 \times \Big(f\Big(\frac{n-1/2}{\Delta}\Big) + \frac{(z-n+1/2)}{\Delta} f'\Big(\frac{n-1/2}{\Delta}\Big)\Big). \tag{8}$$

For any complex number ξ we have $(\sin(\pi\xi)/(\pi\xi))^2 \ll e^{2\pi|\mathrm{Im}\,\xi|}/(1+|\xi|^2)$, and further $f(x) \leqslant 4/x^2$ and $|f'(x)| \leqslant 8/(|x|(4+x^2))$, whence we deduce that

$$\left|G_\Delta(x+iy) + f\left(\frac{1}{\Delta}\right)\right| \ll \frac{\Delta^2}{1+|x+iy|}e^{2\pi|y|}. \tag{9}$$

We now cull from [**3**, Theorem 1.1] various facts about the function $G_\Delta(z)$. This function is entire of exponential type at most 2π, and for real x we have that $G_\Delta(x) \leqslant f_\Delta(x)$. We expect that $G_\Delta(x) + f(1/\Delta)$ is non-negative for all real x, but for our purposes a cruder lower bound suffices. Since $f(x) \geqslant 0$ and $f'(-x) = -f'(x)$, by pairing the terms $n \geqslant 1$ with the terms $1-n \leqslant 0$ we obtain from (8) that

$$\begin{aligned} G_\Delta(x) + f\left(\frac{1}{\Delta}\right) &\geqslant \left(\frac{\cos(\pi x)}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{\Delta} f'\left(\frac{n-1/2}{\Delta}\right)\left(\frac{1}{x-n+1/2} - \frac{1}{x+n-1/2}\right) \\ &= \sum_{n=1}^{\infty}\left(\frac{\sin^2(\pi(x-n+1/2))}{\pi(x^2-(n-1/2)^2)}\right)\frac{2(n-1/2)}{\Delta} f'\left(\frac{n-1/2}{\Delta}\right), \end{aligned}$$

and from this we may easily deduce that there is a constant C such that

$$-C\frac{\Delta^2}{\Delta^2+x^2} \leqslant G_\Delta(x) + f\left(\frac{1}{\Delta}\right) \leqslant f\left(\frac{x}{\Delta}\right). \tag{10}$$

By [**3**, Theorem 1.1(v)] we have

$$\begin{aligned} &\int_{-\infty}^{\infty}\left(f\left(\frac{x}{\Delta}\right) - \left(G_\Delta(x) + f\left(\frac{1}{\Delta}\right)\right)\right) e^{-2\pi i t x}\,dx \\ &\quad = \int_0^\infty \left(\frac{2\lambda}{\lambda^2+4\pi^2t^2} - \hat{L}(\lambda,t)\right)\frac{2(1-\cos(2\Delta\lambda))}{\lambda}\,d\lambda, \end{aligned} \tag{11}$$

where $\hat{L}(\lambda,t) = 0$ if $|t| \geqslant 1$ and for $|t| \leqslant 1$ we have (see [**3**, Lemma 3.2])

$$\hat{L}(\lambda,t) = \frac{(1-|t|)\sinh(\lambda/2)\cos(\pi t) + (\lambda/2\pi)|\sin \pi t|\cosh(\lambda/2)}{\sinh^2(\lambda/2)+\sin^2 \pi t}. \tag{12}$$

Now $f(x/\Delta)$ is integrable, and we may check that

$$\int_{-\infty}^{\infty} f\left(\frac{x}{\Delta}\right) e^{-2\pi i t x}\,dx = \int_0^\infty \frac{2\lambda}{\lambda^2+4\pi^2t^2}\frac{2(1-\cos(2\Delta\lambda))}{\lambda}\,d\lambda,$$

so that $G_\Delta(x) + f(1/\Delta)$ is also integrable and

$$\int_{-\infty}^{\infty}\left(G_\Delta(x) + f\left(\frac{1}{\Delta}\right)\right) e^{-2\pi i x t}\,dx = \int_0^\infty \hat{L}(\lambda,t)\frac{2(1-\cos(2\Delta\lambda))}{\lambda}\,d\lambda. \tag{13}$$

Since

$$\hat{L}(\lambda,t) \ll \frac{1+\lambda}{\sinh(\lambda/2)} + \frac{\lambda}{(\sinh(\lambda/2))^2},$$

and $(1-\cos(2\Delta\lambda))/\lambda \ll \min(1/\lambda, \Delta^2\lambda)$, we deduce from (12) and (13) that

$$\left|\int_{-\infty}^{\infty}\left(G_\Delta(x) + f\left(\frac{1}{\Delta}\right)\right) e^{-2\pi i x t}\,dx\right| \ll \Delta. \tag{14}$$

Moreover from (11) and a little calculus we find that

$$\begin{aligned} \int_{-\infty}^{\infty}(f_\Delta(x) - G_\Delta(x))\,dx &= \int_0^\infty \left(\frac{2}{x} - \frac{1}{\sinh(x/2)}\right)\frac{2(1-\cos(2\Delta x))}{x}\,dx \\ &= 2\log 2 - 2\log(1+e^{-4\pi\Delta}). \end{aligned} \tag{15}$$

We are now in a position to prove Proposition 2.1. We take $g_\Delta(z) = G_\Delta(z\Delta) + f(1/\Delta)$, so that for real x we have $g_\Delta(x) = G_\Delta(x\Delta) + f(1/\Delta) \leqslant f_\Delta(x\Delta) + f(1/\Delta) = f(x)$. Since G_Δ

has exponential type at most 2π, we see that g_Δ has exponential type at most $2\pi\Delta$. Further, $\hat{g}_\Delta(t) = \Delta^{-1}\hat{G}_\Delta(t/\Delta)$. Thus part (i) of Proposition 2.1 follows from (9) and (10), part (ii) from (12)–(14), and part (iii) from (15).

4. *Discussion*

The estimate (5) gives a variant of the main proposition of [**12**] which states that for large t,

$$\log\left|\zeta\left(\frac{1}{2}+it\right)\right| \leqslant \mathrm{Re}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1/2+\lambda/\log x+it}\log n}\frac{\log(x/n)}{\log x}+\frac{(1+\lambda)}{2}\frac{\log t}{\log x}+O\left(\frac{1}{\log x}\right),$$

where $2\leqslant x\leqslant t^2$, and $\lambda\geqslant\lambda_0=0.4912\ldots$ where λ_0 denotes the unique positive real number satisfying $e^{-\lambda_0}=\lambda_0+\lambda_0^2/2$. For large t it is difficult to give good estimates for the sum over n above (or in (5)) and this is the barrier to establishing better estimates for $|\zeta(\frac{1}{2}+it)|$. However one can study the frequency with which such sums get large, and this information is used in [**12**] to understand the size of moments of $\zeta(\frac{1}{2}+it)$.

In light of our work we can view the proposition in [**12**] as constructing a different minorant of our function $f(x)$. We start with (for positive α and x real)

$$K_\Delta(\alpha,x)=2\pi\int_{-\Delta}^{\Delta}\left(1-\frac{|t|}{\Delta}\right)e^{-2\pi\alpha|t|-2\pi itx}\,dt=\frac{2\alpha}{\alpha^2+x^2}-\frac{1}{\pi\Delta}\mathrm{Re}\frac{1-e^{-2\pi\Delta(\alpha+ix)}}{(\alpha+ix)^2}.$$

Integrating both sides from $\alpha_0>0$ to 2, we obtain

$$\int_{\alpha_0}^{2}K_\Delta(\alpha,x)\,d\alpha\leqslant\log\frac{4+x^2}{\alpha_0^2+x^2}+\frac{1}{\pi\Delta}\left(\frac{2}{4+x^2}-\frac{\alpha_0}{\alpha_0^2+x^2}\right)+\frac{1}{\pi\Delta}\int_{\alpha_0}^{2}\frac{e^{-2\pi\Delta\alpha}}{\alpha_0^2+x^2}\,d\alpha,$$

and, upon rearranging,

$$\log\frac{4+x^2}{\alpha_0^2+x^2}\geqslant\int_{\alpha_0}^{2}K_\Delta(\alpha,x)\,d\alpha+\frac{1}{\pi\Delta(\alpha_0^2+x^2)}\left(\alpha_0-\frac{e^{-2\pi\alpha_0\Delta}}{2\pi\Delta}\right)-\frac{1}{\pi\Delta}\frac{2}{4+x^2}.$$

Since $\log((\alpha_0^2+x^2)/x^2)\geqslant\alpha_0^2/(\alpha_0^2+x^2)$, we conclude that

$$f(x)\geqslant\int_{\alpha_0}^{2}K_\Delta(\alpha,x)\,d\alpha+\frac{1}{\alpha_0^2+x^2}\left(\alpha_0^2+\frac{\alpha_0}{\pi\Delta}-\frac{e^{-2\pi\alpha_0\Delta}}{2\pi^2\Delta^2}\right)-\frac{1}{\pi\Delta}\frac{2}{4+x^2}.$$

If we choose $\alpha_0\geqslant\lambda_0/(2\pi\Delta)$, then the middle term above is non-negative and we have shown that for such α_0

$$f(x)\geqslant\int_{\alpha_0}^{2}K_\Delta(\alpha,x)\,d\alpha-\frac{1}{\pi\Delta}\frac{2}{4+x^2}.$$

The first term in the right-hand side above clearly has Fourier transform supported in $[-\Delta,\Delta]$. The second term may be easily approximated by functions having compactly supported Fourier transform. For example, assuming that $2\pi\Delta\geqslant 2$ say, we can see from the definition of K_Δ that $2/(4+x^2)\leqslant\frac{1}{2}K_\Delta(2,x)(1-1/2\pi\Delta)^{-1}$, so that $f(x)\geqslant\int_{\alpha_0}^{2}K_\Delta(\alpha,x)\,d\alpha-K_\Delta(2,x)/(2\pi\Delta-1)$. Using the explicit formula with such a minorant gives an alternative proof of the proposition in [**12**]. The construction of minorants given above amounts to taking convolutions with functions whose Fourier transforms have compact support. Although this is not optimal, the method works for related functions such as $\log((4+x^2)/(\alpha^2+x^2))$ (this arises in bounding $\log|\zeta(\frac{1}{2}+\alpha+it)|$) which do not fit the framework of Carneiro and Vaaler. However recent work of Carneiro, Littmann, and Vaaler [**2**] develops a new method which allows one to find optimal minorants for the function $\log((4+x^2)/(\alpha^2+x^2))$, and using this Carneiro and Chandee [**1**] have established good estimates for $\log|\zeta(\frac{1}{2}+\alpha+it)|$.

Theorem 1.1 may be extended to general L-functions. To be concrete, consider the framework described in [**9**, Chapter 5]. Thus we consider L-functions given in $\mathrm{Re}\,s>1$ by the absolutely convergent series and product

$$L(f,s)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p\prod_{j=1}^{d}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where the 'degree' d is a fixed natural number. We assume that there is an integer $q(f)\geqslant 1$ and complex numbers κ_j with $\mathrm{Re}(\kappa_j)>-1$ such that

$$\Lambda(f,s)=\left(\frac{q(f)}{\pi^d}\right)^{s/2}\prod_{j=1}^{d}\Gamma\left(\frac{s+\kappa_j}{2}\right)L(f,s)$$

is entire of order 1 except possibly for poles at $s=0$ and 1. Moreover, we suppose that a functional equation

$$\Lambda(f,s)=\epsilon(f)\Lambda(\overline{f},1-s),$$

holds, where $\epsilon(f)$ is a complex number of size 1, and $\Lambda(\bar{f},s)=\overline{\Lambda(f,\bar{s})}$. We assume the Generalized Riemann Hypothesis for $L(f,s)$, namely that the zeros of $\Lambda(f,s)$ all lie on the line $\mathrm{Re}(s)=\frac{1}{2}$, and then seek a bound for $L(f,\frac{1}{2})$ in terms of the *analytic conductor* $C(f):=q(f)\prod_{j=1}^{d}(3+|\kappa_j|)$. Making minor modifications to our argument we find that

$$\log\left|L\left(f,\frac{1}{2}\right)\right|\leqslant\frac{\log C(f)}{2\pi\Delta}\log\left(\frac{2}{1+e^{-4\pi\Delta}}\right)+\frac{1}{4\pi}\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}}g_\Delta\left(\frac{\log n}{2\pi}\right)(\Lambda_f(n)+\Lambda_{\bar{f}}(n))$$
$$+O\left(\frac{\Delta^2e^{\pi\Delta}}{1+\Delta C(f)}+1\right), \qquad (16)$$

where $\Lambda_f(n)$ and $\Lambda_{\bar{f}}(n)$ are the Dirichlet series coefficients of $-L'/L(f,s)$ and $-L'/L(\bar{f},s)$ respectively. If we now assume the Ramanujan conjectures (which imply that $|\Lambda_f(n)|\leqslant d\Lambda(n)$) then, choosing $\pi\Delta=(1-o(1))\log\log C(f)$ and estimating the sum over n in (16) trivially, we obtain that

$$\log\left|L\left(f,\frac{1}{2}\right)\right|\leqslant\left(\frac{\log 2}{2}+o(1)\right)\frac{\log C(f)}{\log\log C(f)},$$

which is the analog of Theorem 1.1. If we do not assume the Ramanujan conjectures, then using that $|\Lambda_f(n)|\leqslant dn\Lambda(n)$ (which follows from our assumption that the Euler product converges absolutely in $\mathrm{Re}(s)>1$), and choosing $\pi\Delta=(\frac{1}{3}-o(1))\log\log C(f)$, we obtain

$$\log\left|L\left(f,\frac{1}{2}\right)\right|\leqslant 3\left(\frac{\log 2}{2}+o(1)\right)\frac{\log C(f)}{\log\log C(f)}.$$

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