Advanced topics in the study of the Riemann zeta function - NTNU 2021
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PROBLEM SET 1

Let $\Omega \subset \mathbb{C}$ be an open set.

(1) Let $\{f_n\}_{n\geq 1}$ be a sequence of analytic function in Ω such that f_n converges uniformly in compacts of Ω to f. Prove that f is analytic in Ω. Moreover, prove that f_n' converges uniformly in compacts of Ω to f'.

(2) Prove that for $\sigma > 1$ we have
$$\frac{1}{\sigma-1} \leq \zeta(\sigma) \leq \frac{\sigma}{\sigma-1}.$$
In particular, show that $\lim\limits_{\sigma\to 1^+} \zeta(\sigma) = +\infty$.

(3) Show that $\lim\limits_{\sigma\to +\infty} \zeta(\sigma) = 1$.

(4) The values of $\zeta(s)$ and $\eta(s)$:
 (a) Find the values of $\zeta(2)$ and $\zeta(4)$.
 (b) Find the values of $\eta(1)$, $\eta(2)$ and $\eta(4)$.

Give proofs to find these values (using Fourier analysis for instance).

(5) Prove Abel's identity: Let a_n be a sequence of complex numbers., and define $A : (0,\infty) \to \mathbb{C}$ by
$$A(x) = \sum_{n\leq x} a_n,$$
and $A(x) = 0$ if $0 < x < 1$. Assume f has a continuous derivative on the interval $[y, x]$ where $0 < y < x$. Then we have
$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)\,\mathrm{d}t.$$
Hint: Use the fact that $A(n) - A(n-1) = a_n$, to prove that
$$\sum_{y<n\leq x} a_n f(n) = \sum_{n=[y]+1}^{[x]-1} A(n)\big\{f(n) - f(n+1)\big\} + A([x])f([x]) - A([y])f([y]+1).$$
Then, use the fundamental calculus theorem and the fact that $A(t)$ is constant in $[n, n+1)$ to obtain the desired result.

(6) Using Abel's identity, deduce Euler's summation formula: if f has a continuous derivative on the interval $[y, x]$ where $0 < y < x$ then
$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,\mathrm{d}t + \int_y^x (t-[t])f'(t)\,\mathrm{d}t + f(x)([x]-x) - f(y)([y]-y).$$

(7) For $x \geq 2$, prove the following estimates:

(a) $\displaystyle\sum_{n\leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right)$, for some constant $C \in \mathbb{R}$.

(b) $\displaystyle\sum_{n\leq x} \frac{1}{n^\sigma} = \frac{x^{1-\sigma}}{1-\sigma} + \zeta(\sigma) + O(x^{-\sigma})$, for $\sigma > 0$, $\sigma \neq 1$.

(c) $\displaystyle\sum_{n> x} \frac{1}{n^\sigma} = O(x^{1-\sigma})$, for $\sigma > 1$.

(d) $\displaystyle\sum_{n\leq x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O(x^\alpha)$, for $\alpha \geq 0$.

(e) $\displaystyle\sum_{n\leq x} \frac{\log n}{n} = \frac{\log^2 x}{2} + A + O\left(\frac{\log x}{x}\right)$, for some constant $A \in \mathbb{R}$.

(f) $\displaystyle\sum_{n\leq x} \frac{1}{n \log n} = \log\log x + B + O\left(\frac{1}{x \log x}\right)$, for some constant $B \in \mathbb{R}$.

(8) Prove that, uniformly in compacts of $\mathrm{Re}\, s > 1$, we have the convergence

$$\sum_p \log\left(1 - \frac{1}{p^s}\right) = -\log\zeta(s).$$

Here the logarithm of $\zeta(s)$ is defined such that $\log\zeta(2) \in \mathbb{R}$.

(9) Define the Möbius function as follows:

$$\mu(n) = \begin{cases} 1, & if\, n = 1, \\ (-1)^k, & if\ \ n = p_1 p_2 \cdots p_k \text{ (n is a product of k distinct primes)} \\ 0, & \text{otherwise.} \end{cases}$$

Then, prove that for $n > 1$ we have

$$\sum_{d|n} \mu(d) = 0,$$

where the sum runs over the positive divisors d of n. Note that $\sum_{d|1} \mu(d) = \mu(1) = 1$.

(10) Let $F(s)$ and $G(s)$ be two functions with representation

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \quad \text{and} \quad G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

for some sequences $f(n)$ and $g(n)$. Suppose that F converges absolutely for $\mathrm{Re}\, s > a$ and G converges absolutely for $\mathrm{Re}\, s > b$. Prove that, in $\mathrm{Re}\, s > \max\{a, b\}$ we have the representation

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

where this series converges absolutely, and

$$h(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

(11) Prove that, for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

Hint: Use problems (9) and (10).

(12) Prove that $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.

(13) Prove that for $\sigma > 0$ ($\sigma \neq 1$) we have

$$\frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{\sigma}{\sigma - 1}.$$

Note that this is a refination of the problem (2).

(14) Let $\mathcal{M}(\mathbb{R})$ the family of moderate decrease functions. This means that $f \in \mathcal{M}(\mathbb{R})$, if f is continuous on $\mathbb{R}$ and for some $M > 0$

$$|f(x)| \leq \frac{M}{1 + x^2}, \text{ for all x} \in \mathbb{R}.$$

Denote by $\widehat{f}$ the Fourier transform of f defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, \mathrm{d}x,$$

for all $\xi \in \mathbb{R}$. Prove the classical Poisson summation formula: if $f \in \mathcal{M}(\mathbb{R})$ and $\widehat{f} \in \mathcal{M}(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x + n) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{2\pi i k x},$$

for all $x \in \mathbb{R}$.

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