Advanced topics in the study of the Riemann zeta function - NTNU 2021
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PROBLEM SET 2

(15) For $\sigma > 1$, prove that

$$\left|\frac{1}{\zeta(s)}\right| \leq \frac{\zeta(\sigma)}{\zeta(2\sigma)}.$$

Hint: Use exercise (11).

(16) For each $m \geq 1$ a natural number, we want to obtain a expression for

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}},$$

(a) Apply the Poisson summation formula to obtain, for $t > 0$ that:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi t|n|}.$$

(b) Prove the following identity valid for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}.$$

(c) Use the fact that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}z^{2m},$$

to deduce the formula

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

The numbers B_{2m} are known as the Bernoulli numbers. Compute $\zeta(6)$ and $\zeta(8)$.

(17) The following facts have been given in class as exercises: Define the functions

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi x n^2} \quad \text{and} \quad \omega(x) = \sum_{n=1}^{\infty} e^{-\pi x n^2},$$

for $x > 0$. Using Poisson summation formula, one can prove the modularity of θ:

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\,\theta(x).$$

(a) Prove that θ and ω are continuous functions on $(0, \infty)$.

(b) Using the modularity of θ, prove that there are constants $C > 0$ and $D > 0$ such that

$$0 < \omega(x) \leq Ce^{-\pi x}, \quad \text{for } x > 1,$$

$$0 < \omega(x) \leq Dx^{-1/2}, \quad \text{for } 0 < x \leq 1,$$

(c) Justify the step: For $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi x n^2}x^{s/2-1}\,\mathrm{d}x = \int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi x n^2}x^{s/2-1}\,\mathrm{d}x.$$

(d) Prove that $g : \mathbb{C} \to \mathbb{C}$ is an entire function, where

$$g(s) = \int_1^\infty \omega(x)\left(x^{s/2} + x^{(1-s)/2}\right) \frac{\mathrm{d}x}{x}.$$

(18) Find the zeros of the Dirichlet eta function $\eta(s)$ in $\operatorname{Re} s \geq 1$ and $\operatorname{Re} s \leq 0$.

(19) Compute the values of $\zeta(-1)$ and $\dfrac{\zeta'}{\zeta}(0)$.

(20) Let

$$\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = e^{A+Bs}\prod_\rho \left(1 - \frac{s}{\rho}\right)e^{s/\rho}.$$

(a) Find the values of A and B.

(b) Show the following version of the functional equation:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s).$$

(21) Show that if k is a positive integer, then

$$\zeta'(-2k) = \frac{(-1)^k\,(2k)!\,\zeta(2k+1)}{2^{2k+1}\pi^{2k}}.$$

(22) Stirling's formula for the Gamma function.

(a) For a fixed $\delta > 0$ and $-\pi + \delta < \arg(s) < \pi - \delta$, show that

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O(|s|^{-1}),$$

as $|s| \to \infty$.

(b) Under the same conditions:

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(|s|^{-1}),$$

as $|s| \to \infty$.

(23) Prove that

$$\sum_\rho \frac{1}{|\rho|} = \infty,$$

where the sum runs over the non-trivial zeros of $\zeta(s)$.
Hint: Use the fact that $|1-w| \leq e^{|w|}$ for all $w \in \mathbb{C}$.

(24) Assume that $\alpha, \beta \in \mathbb{R}$ such that $\beta^2 \leq 8(\alpha - 1)$. Then, prove that, for $\sigma > 1$ we have

$$|\zeta(\sigma)|^\alpha|\zeta(\sigma+it)|^\beta|\zeta(\sigma+2it)| \geq 1, \tag{0.1}$$

(25) Assume that $4 \leq \beta \leq \alpha + 1$. Prove that, for $\sigma > 1$ the inequality (0.1) holds.

(26) Let $w(s)$ be an entire function. Let $w^*(s)$ be the entire function defined as $w*(s) = \overline{w(\overline{s})}$. Suppose that for $s \in \mathbb{C}$ with $\operatorname{Im} s > 0$ we have

$$|w(s)| < |w^*(s)|.$$

Show that all the zeros of the function $w(s) + w^*(s)$ are real. In particular, show that if f is an entire function such that

$$f(s) = s^m e^{As+B} \prod_{n=1}^{p} \left(1 - \frac{s}{a_n}\right) e^{s/a_n},$$

where $m \geq 0$ is an integer, $A, B \in \mathbb{R}$ and $a_n \in \mathbb{R}$, $p \geq 1$ or $p = \infty$, and $\sum_{n=1}^{\infty} |a_n|^{-2} < \infty$, then for any $a > 0$ and $b \in \mathbb{R}$ we have that the function

$$G(z) = f(z - ia)e^{-ib} + f(z + ia)e^{ib}$$

only has real zeros.

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