Advanced topics in the study of the Riemann zeta function - NTNU 2021
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PROBLEM SET 3

(27) Let γ be given and define $f(\beta) = \beta/(\beta^2+\gamma^2)$. Show that if $0 \leq \beta \leq 1$, then $f(\beta) \geq \beta/(1+\gamma^2)$. Deduce that if $0 \leq \beta \leq 1$, then $f(\beta) + f(1-\beta) \geq f(0) + f(1)$. Use this to prove that if $\rho = \beta + i\gamma$ is a non-trivial zero of $\zeta(s)$ with $\beta \neq 1/2$, then

$$|\gamma| \geq \left(-\frac{2}{B} - 1\right)^{1/2} = 9.2518...,$$

where B is the parameter in the Hadamard factorization of $\xi(s)$.

(28) (a) Show that

$$\sum_{\rho} \operatorname{Re} \frac{1}{\sigma - \rho} = \frac{1}{2}\log\sigma + O(1),$$

for $\sigma \geq 2$, where the sum is over all non-trivial zeros of $\zeta(s)$.

(b) Deduce that

$$\sum_{\rho} \left(\operatorname{Re}\frac{1}{\sigma-\rho} - \frac{3}{4}\operatorname{Re}\frac{1}{2\sigma-\rho} \right) = \frac{1}{8}\log\sigma + O(1),$$

for $\sigma \geq 2$.

(c) Show that each summand above is $\leq 1/(\sigma-1)$.

(d) Show that if $|\gamma| \geq 3\sigma$ and σ large, then the summand arising from ρ in the sum above is ≤ 0.

(e) Conclude that $N(T) \gg T\log T$, when T is large.

(29) Put

$$f(s) = \operatorname{Re}\left\{ \frac{1}{s+1} - \frac{3}{4(s+2)} \right\}.$$

(a) Show that if $t \geq 2$, then

$$\sum_{\rho} f(1+it-\rho) = \frac{1}{8}\log t + O(1),$$

where the sum is over all non-trivial zeros of $\zeta(s)$.

(b) Show that $f(s) \leq 1$, where $s = \sigma + it$ with $\sigma \geq 0$.

(c) Show that if $0 \leq \sigma < 2$, then $f(s) \leq 0$ when

$$t^2 \geq \frac{(\sigma+1)(\sigma+2)(\sigma+5)}{2-\sigma}.$$

(d) Deduce that $f(s) \leq 0$ if $0 < \sigma < 1$ and $|t| \geq 6$.

(e) Show that $N(T+6) - N(T-6) \gg \log T$ for all T sufficiently large.

(30) Prove that for $T \geq 2$ we have

$$\sum_{|\gamma| \leq T} \frac{1}{|\gamma|} = O(\log^2 T), \quad \text{and} \quad \sum_{|\gamma| > T} \frac{1}{\gamma^2} = O\left(\frac{\log T}{T}\right),$$

where the sum runs over the imaginary part γ of the non-trivial zeros of $\zeta(s)$.

(31) Let $0 < \gamma_1 \leq \gamma_2 \leq ...$ be the consecutive ordinates of the non-trivial zeros of $\zeta(s)$ with positive imaginary parts. Prove that

$$\gamma_n \sim \frac{2\pi n}{\log n}, \quad \text{as} \quad n \to \infty.$$

(32) Let $N(T)$ be the counting function of the non-trivial zeros of $\zeta(s)$ with $0 < \gamma \leq T$. Decide the veracity of the following estimates:

(a) $N(T^2 + T) - N(T^2) \leq N(3T) - N(T)$, for T sufficiently large.

(b) $N(T^2 + T) - N(T^2) \leq N(4T) - N(2T)$, for T sufficiently large.

(33) Use Proposition of the class 7, to prove that, if the Riemann hypothesis is true, for $\frac{1}{2} < \sigma \leq 2$ and $t \geq 2$ we have

$$\frac{\zeta'}{\zeta}(\sigma + it) = O\left(\frac{\log t}{\sigma - \frac{1}{2}}\right).$$

(34) Prove that for any integer $k \geq 1$, we have for $s = \sigma + it$ with $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$ and $t \geq 2$

$$\sum_{|\gamma - t| > 1} \frac{1}{(s-\rho)^{k+1}} = O(\log t).$$

(35) $0 < \gamma_1 \leq \gamma_2 \leq ...$ be the consecutive ordinates of the non-trivial zeros of $\zeta(s)$ with positive imaginary parts. Show that

$$\gamma_{n+1} - \gamma_n = O(1).$$

Hint: Find $H > 0$ such that $N(T + H) - N(T) > 0$.

(36) Prove that there is $C > 0$ and $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$ there is $T_n \in \mathbb{R}$ with $n \leq T_n \leq n + 1$ such that

$$|\gamma - T_n| \geq \frac{C}{\log T_n},$$

for all $\gamma \in \mathbb{R}$, where γ is the ordinate of a non-trivial zero of $\zeta(s)$.

(37) *Let $S(t)$ be the argument function defined in class:

$$S(t) = \frac{1}{\pi} \arg \zeta(\tfrac{1}{2} + it),$$

for $t > 0$.

(a) Prove that

$$S(t) = -\frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \operatorname{Im} \frac{\zeta'}{\zeta}(\sigma + it) \, d\sigma.$$

(b) Define for $T > 0$:

$$S_1(T) = \int_0^T S(t) \, dt.$$

Prove that

$$S_1(T) = \frac{1}{\pi} \int_{\frac{1}{2}}^{2} \log |\zeta(\sigma + iT)| \, d\sigma + O(1).$$

(c) Prove that for $T \geq 2$:

$$S_1(T) = O(\log T).$$

Hint: Take a look on Titchmarsh's book Chapter 9.

(38) A refined version of the Riemann-von Mangoldt formula gives, for $T \geq 2$:

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + \frac{1}{48\pi T} + O\left(\frac{1}{T^3}\right).$$

Define the function

$$N_1(T) = \sum_{0<\gamma\leq T} \gamma,$$

where the sum runs over the imaginary part γ of the non-trivial zeros of $\zeta(s)$. Use the refined version of the Riemann-von Mangoldt formula to prove that

$$N_1(T) = \frac{T^2}{4\pi}\log\frac{T}{2\pi} - \frac{T^2}{8\pi} - \frac{\log T}{48\pi} + TS(T) - S_1(T) + O(1).$$

(39) *Prove the second mean value theorem for integrals: Let $G : [a,b] \to \mathbb{R}$ be a monotonic function and $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Then, there is $c \in (a,b]$ such that

$$\int_a^b G(x)f(x)\mathrm{d}x = G(a^+)\int_a^c f(x)\mathrm{d}x + G(b^-)\int_c^b f(x)\mathrm{d}x.$$

(40) *Define the sawtooth function $\psi : \mathbb{R} \to \mathbb{R}$ as

$$\psi(x) = \begin{cases} x - [x] - \frac{1}{2}, & if\, x \notin \mathbb{Z} \\ 0 & if\, x \in \mathbb{Z}. \end{cases}$$

Prove that

$$\psi(x) = -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\sin 2\pi m x}{m},$$

for all $x \in \mathbb{R}$, and the partial sums of its Fourier series are uniformly bounded.

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