Advanced topics in the study of the Riemann zeta function - NTNU 2021
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PROBLEM SET 4

(41) Prove that for any $m, n \in \mathbb{Z}$ such that $m \neq n$, we have

$$\int_{-\infty}^{\infty} \frac{\sin^2(\pi x)}{(x-n)(x-m)} \mathrm{d}x = 0.$$

Conclude that $\left\{\frac{\sin(x-n)}{x-n}\right\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R})$.

(42) Define the function $\psi(x) = F(x) - x_0^+(x)$, where

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\, \mathrm{d}s.$$

Following the ideas developed in class, prove that for $|t| \geq 1$ we have:

$$\widehat{\psi}(t) = -\frac{1}{2\pi i t}.$$

(43) Let $\frac{1}{2} < \sigma \leq 1$. Find the asymptotic formulas for

$$\int_1^T |\zeta(\sigma + it)|^2 \,\mathrm{d}t.$$

(44) A classical theorem in the theory of Dirichlet series establishes the following:
Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that Ω contains the strip $\sigma_1 \leq \operatorname{Re} s \leq \sigma_2$. Suppose that $f(\sigma + it) = O\big(e^{\varepsilon|t|}\big)$ in the strip $\sigma_1 \leq \operatorname{Re} s \leq \sigma_2$, for every $\varepsilon > 0$. Suppose that $f(\sigma_1 + it) = O\big(|t|^{k_1}\big)$ and $f(\sigma_2 + it) = O\big(|t|^{k_2}\big)$. Then, we have

$$f(\sigma + it) = O\big(|t|^{k(\sigma)}\big),$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$, where $k(\sigma)$ is the linear function of σ which takes the values k_1 and k_2 for $\sigma = \sigma_1$ and $\sigma = \sigma_2$ respectively.
Hint: See the book *The theory of functions* by E. C. Titchmarsh, p. 180-181, to see an easy proof.

(45) Let $\Delta > 0$. Prove that

$$\int_0^{\infty} \left\{e^{-\lambda|x|} - e^{-\lambda}\right\} \frac{2(1-\cos(2\Delta\lambda))}{\lambda} \mathrm{d}\lambda = \log\left(\frac{4\Delta^2 + x^2}{x^2}\right) - \log(4\Delta^2 + 1),$$

for all $x \in \mathbb{R}$.

(46) Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the function

$$f(x) = \log\left(\frac{4+x^2}{x^2}\right).$$

Prove the Proposition mentioned in class: Let $\Delta \geq 1$. Then, there is an even entire function $m_\Delta : \mathbb{C} \to \mathbb{C}$ such that:

(a) There is a constant $C > 0$ such that for all $x \in \mathbb{R}$:

$$\frac{-C}{1+x^2} \le m_\Delta(x) \le f(x).$$

(b) For all $z \in \mathbb{C}$ we have:

$$|m_\Delta(z)| \ll \frac{\Delta^2}{1+\Delta|z|} e^{2\pi\Delta|\mathrm{Im}\, z|} \quad \text{for all } z \in \mathbb{C}.$$

(c) $m_\Delta \in L^1(\mathbb{R})$, $\widehat{m_\Delta}(\xi) = 0$ for $|\xi| \ge \Delta$, and $\widehat{m_\Delta}(\xi) = O(1)$.

(d) The distance $L^1(\mathbb{R})$ is given by:

$$\int_{-\infty}^{\infty} \big\{ f(x) - m_\Delta(x) \big\} \mathrm{d}x = \frac{1}{\Delta}\Big(2\log 2 - 2\log(1 + e^{-4\pi\Delta}) \Big).$$

(e) When $|\mathrm{Im}\, z| \le \frac{1}{2} + \varepsilon$ and $|\mathrm{Re}\, z| \to \infty$ we have

$$\big| m_\Delta(z)(1+|z|)^2 \big| \ll_\Delta 1.$$

(47) (Chebychev) We want to prove that there is a constant $C > 0$ such that, for $x \ge 2$ we have

$$\psi(x) := \sum_{n \le x} \Lambda(n) \le C\, x. \tag{0.1}$$

(a) Define the function $L(x) = \sum_{n \le x} \log(n)$. Prove that $L(x) = x\log x - x + O(\log x)$.

(b) Prove that

$$\log n = \sum_{d|n} \Lambda(d) = \sum_{d|n} \Lambda\left(\frac{n}{d}\right).$$

(c) Prove that

$$L(x) = \sum_{d \le x} \psi\left(\frac{x}{d}\right)$$

(d) Prove that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le L(x) - 2L\left(\frac{x}{2}\right) \le \psi(x).$$

(e) Prove that

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le x\log 2 + O(\log x).$$

(f) Conclude (0.1).

(48) Prove Hadamard's three-circles theorem: Let $\Omega \subset \mathbb{C}$ be an open set such that contains the annulus $r_1 \le |z| \le r_3$. Let $f : \Omega \to \mathbb{C}$ be an holomorphic function. Define $M(r)$ the maximum of $|f(z)|$ on the circle $|z| = r$. Then, for $r_1 < r_2 < r_3$ we have:

$$(M_2)^{\log(r_3/r_1)} \le (M_1)^{\log(r_3/r_2)} (M_3)^{\log(r_2/r_1)}.$$

(49) Prove Borel-Carathéodory theorem: Let $\Omega \subset \mathbb{C}$ be an open set such that contains the disc $|z| \le R$. Then, for $0 < r < R$ we have that

$$\max_{|z| \le r} |f(z)| \le \frac{2r}{R-r} \max_{|z|=R} \mathrm{Re}\, f(z) + \frac{R+r}{R-r} |f(0)|.$$

(50) Let $\{a_n\}_{n=1}^N$ be complex numbers. Then, we have for $T \geq 2$:

$$\int_0^T \left| \sum_{n=1}^N a_n n^{it} \right|^2 \mathrm{d}t = (T + O(N)) \sum_{n=1}^N |a_n|^2.$$

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