$\frac{\zeta'}{\zeta}(0)$

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We start from the functional equation

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\zeta(s)$$

or more so its logarithmic derivative:

$$-\frac{\zeta'}{\zeta}(1-s) = -\log(2\pi) - \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right) + \frac{\Gamma'}{\Gamma}(s) + \frac{\zeta'}{\zeta}(s)$$

We want to take the limit as $s \to 1$, but this seems troubleful as the RHS might explode. Luckily the explosion happens in "both directions" and cancel out each other. Taking the limit we get

$$-\frac{\zeta'}{\zeta}(0) = -\log(2\pi) + \frac{\Gamma'}{\Gamma}(1) + \lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) - \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right)\right)$$

For ζ'/ζ and tan we have the following Laurent series valid on some annulus around $s = 1$:

$$\frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{n=0}^{\infty} a_n(s-1)^n \qquad \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right) = -\frac{1}{s-1} + \frac{\pi^2}{12}(s-1) + O(|s-1|^3)$$

Moving $-\frac{1}{s-1}$ over to the other side in both expressions we obtain formulas that are valid in some neigborhood of $s = 1$. We can then calculate the limit:

$$\begin{aligned}
\lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) - \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right)\right) &= \lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) - \frac{\pi}{2}\tan\left(\frac{\pi s}{2}\right) + \underbrace{\frac{1}{s-1} - \frac{1}{s-1}}_{=0}\right) \\
&= \lim_{s\to 1}\left(\sum_{n=0}^{\infty} a_n(s-1)^n - \frac{\pi^2}{12}(s-1) + O(|s-1|^3)\right) \\
&= a_0
\end{aligned}$$

By the Laurent series we also have

$$a_0 = \lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right)$$

In class we proved that ζ has the following analytic continuation for $\mathrm{Re}(s) > 0$:

$$\zeta(s) = 1 + \frac{1}{s-1} + s\int_1^\infty \frac{[t]-t}{t^{s+1}}\,\mathrm{d}t = \frac{s}{s-1} + s\mathcal{J}(s)$$

where $\mathcal{J}(s)$ is just shorthand notation for the integral. We thus get that

$$\zeta(s)(s-1) = s + s(s-1)\mathcal{J}(s)$$

Logarithmically differentiating gives

$$\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1} = \frac{1 + (2s-1)\mathcal{J}(s) + (s^2-s)\mathcal{J}'(s)}{s + s(s-1)\mathcal{J}(s)}$$

Taking the limit as $s \to 1$ we arrive at

$$\lim_{s\to 1}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right) = 1 + \mathcal{J}(1)$$

Calculating $\mathcal{J}(1)$ is not very hard.

$$\begin{aligned}
\mathcal{J}(1) &= \lim_{N\to\infty}\int_1^N \frac{[t]-t}{t^2}\,\mathrm{d}t \\
&= \lim_{N\to\infty}\left(\int_1^N \frac{[t]}{t^2}\,\mathrm{d}t - \int_1^N \frac{1}{t}\,\mathrm{d}t\right) \\
&= \lim_{N\to\infty}\left(\sum_{n=1}^N \int_n^{n+1} \frac{n}{t^2}\,\mathrm{d}t - \log N\right) \\
&= \lim_{N\to\infty}\left(\sum_{n=1}^N \frac{1}{n+1} - \log N\right) \\
&= -1 + \gamma
\end{aligned}$$

where the very last equality follows from the definition of the Euler-Mascheroni constant. We use the Weierstrass product for Γ to evaluate the Γ term. The Weiestrass product for Γ is

$$\Gamma(s) = \frac{e^{-\gamma s}}{s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)^{-1} e^{s/n}$$

Taking the logarithmic derivative we get

$$\frac{\Gamma'}{\Gamma}(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^\infty \left(-\frac{1}{n}\frac{n}{n+s} + \frac{1}{n}\right) = -\gamma - \frac{1}{s} + \sum_{n=1}^\infty \left(-\frac{1}{n+s} + \frac{1}{n}\right)$$

When evaluated in $s = 1$ we get a telescoping series and thus

$$\frac{\Gamma'}{\Gamma}(1) = -\gamma - 1 + \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) = -\gamma$$

We can finally conclude that

$$\frac{\zeta'}{\zeta}(0) = -\left(-\log(2\pi) - \gamma + \gamma\right) = \log(2\pi)$$