Class 11: Beurling-Selberg problem and Hilbert's inequality

Andrés Chirre
Norwegian University of Science and Technology - NTNU

11-October-2021

Review:

Review:

1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Review:

1. For $\mathrm{Re}\, s > 1$ we define the Riemann zeta-function $\zeta(s)$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

2. For $\mathrm{Re}\, s > 0$,

$$\zeta(s) = 1 + \frac{1}{s-1} + s \int_1^{\infty} \frac{[t] - t}{t^{s+1}} \mathrm{d}t.$$

Theorem (Approximation formula)

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ and $|t| < \frac{2\pi x}{C}$.

Proposition

Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function such that $f'(x)$ is monotone and $|f'(x)| \leq \delta < 1$. Then

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \mathrm{d}x + O\left(\frac{1}{1-\delta}\right).$$

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \leq \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi}\sum_{n\geq 2} \frac{\Lambda(n)}{\sqrt{n}}\left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Review Fourier Analysis

Definition

Let $p \geq 1$. The vector space $L^p(\mathbb{R})$ is defined by

$$L^p(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x < \infty \right\},$$

where $\mathrm{d}x$: denotes the Lebesgue measure.
This vector space has norm

$$||f||_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \mathrm{d}x \right)^{1/p}.$$

The Fourier transform in $L^1(\mathbb{R})$

Definition

Let $f \in L^1(\mathbb{R})$. We define the Fourier transform of f as

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} \mathrm{d}x,$$

for $t \in \mathbb{R}$.

Example

1. Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1, 1]$. Then

$$\widehat{\chi_{[-1,1]}}(t) = \frac{\sin 2\pi t}{\pi t}.$$

Example

1. Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1,1]$. Then
$$\widehat{\chi_{[-1,1]}}(t) = \frac{\sin 2\pi t}{\pi t}.$$

2. The function $f(x) = \text{máx}\{1-|x|, 0\}$ has
$$\widehat{f}(t) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

Example

1. Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1,1]$. Then
$$\widehat{\chi_{[-1,1]}}(t) = \frac{\sin 2\pi t}{\pi t}.$$

2. The function $f(x) = \text{máx}\{1-|x|, 0\}$ has
$$\widehat{f}(t) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$

3. The function $g(x) = \dfrac{1}{\pi(1+x^2)}$ has
$$\widehat{g}(t) = e^{-2\pi|t|}.$$

Proposition

1. *For all $t \in \mathbb{R}$ we have*

$$|\widehat{f}(t)| \leq \|f\|_1.$$

2. *The function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is continuous.*
3. *Riemann-Lebesgue lemma:*

$$\lim_{t \to \pm\infty} \widehat{f}(t) = 0.$$

The Fourier transform in $L^2(\mathbb{R})$

Proposition

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\|f\|_2 = \|\widehat{f}\|_2$. In particular, the operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is linear, isometric and surjective.

The Fourier transform in $L^2(\mathbb{R})$

Proposition

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\|f\|_2 = \|\widehat{f}\|_2$. In particular, the operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is linear, isometric and surjective.

Example

The function $L(x) = \dfrac{\sin \pi x}{\pi x}$ satisfies

$$\widehat{L}(t) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(t).$$

Poisson summation formula

Theorem

Let $f \in L^1(\mathbb{R})$ *be a normalized function of bounded variation. Then*

$$\sum_{m\in\mathbb{Z}} f(x+m) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

for all $x \in \mathbb{R}$.

Poisson summation formula

Theorem

Let $f \in L^1(\mathbb{R})$ be a normalized function of bounded variation. Then

$$\sum_{m \in \mathbb{Z}} f(x+m) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k x}$$

for all $x \in \mathbb{R}$.

Theorem

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function such that $|f(x)| \ll 1/(1+|x|)^{1+\delta}$ and $|\widehat{f}(x)| \ll 1/(1+|x|)^{1+\delta}$. Then

$$\sum_{m \in \mathbb{Z}} f(x+m) = \sum_{k \in \mathbb{Z}} \widehat{f}(k) e^{2\pi i k x}$$

for all $x \in \mathbb{R}$.

Entire functions of exponential type

Definition

We say that an entire function has exponential type $2\pi\delta$*, if for any* $\varepsilon > 0$ *there is* $C_\varepsilon > 0$ *such that*

$$|f(z)| \leq C_\varepsilon e^{(2\pi\delta+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$.

Example

1. The function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has exponential type 0.

Example

1. The function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has exponential type 0.
2. The function $E(z) = e^{2\pi\delta z}$ has exponential type $2\pi\delta$.

Example

1. The function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has exponential type 0.
2. The function $E(z) = e^{2\pi\delta z}$ has exponential type $2\pi\delta$.
3. The function $L(z) = \sin(\pi z)$ has exponential type π.

Example

1. The function $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ has exponential type 0.
2. The function $E(z) = e^{2\pi\delta z}$ has exponential type $2\pi\delta$.
3. The function $L(z) = \sin(\pi z)$ has exponential type π.
4. The function $G(z) = \left(\frac{\sin \pi z}{\pi z}\right)^2$ has exponential type 2π.

Paley-Wiener theorem

Theorem

Let F be an entire function such that $F \in L^2(\mathbb{R})$. The following statements are equivalent:

1. *F has exponential type $2\pi\delta$.*
2. *$\widehat{F}$ has compact support in $[-\delta, \delta]$ a.e.*

Example

1. $L(z) = \dfrac{\sin \pi z}{\pi z}$ has exponential type π and satisfies

$$\widehat{L}(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t).$$

2. $G(z) = \left(\dfrac{\sin \pi z}{\pi z}\right)^2$ has exponential type 2π and satisfies

$$\widehat{G}(t) = \text{máx}\{1 - |t|, 0\}.$$

Interpolation formulas

Theorem (Shannon-Whittaker)

Let F be an entire function of exponential type π such that $F \in L^2(\mathbb{R})$. Then

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \frac{F(n)}{z-n},$$

where the series converges uniformly in compacts of $\mathbb{C}$.

Interpolation formulas

Theorem (Vaaler, 1985)

Let F be an entire function of exponential type 2π such that $F \in L^2(\mathbb{R})$. Then

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{F(n)}{(z-n)^2} + \sum_{n\in\mathbb{Z}} \frac{F'(n)}{(z-n)} \right\},$$

where the series converges uniformly in compacts of $\mathbb{C}$.

Beurling-Selberg problem

Delta problem

Let $\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$\delta(x) = \begin{cases} 1 & \textit{if} \quad x = 0 \\ 0 & \textit{if} \quad x \neq 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \delta(x)$ for all $x \in \mathbb{R}$.

Find:

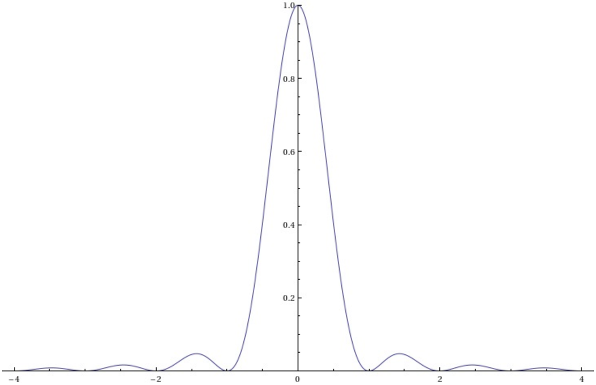
$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \delta(x)\big) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} f(x) \mathrm{d}x = 1.$$

$$F(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2.$$

$$F(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$



Delta problem

$\Rightarrow$

Pair correlation of the zeros of the Riemann zeta-function, multiplicity of the zeros of zeta.

Beurling's problem - 1930's

Arne Carl-August Beurling (1905 - 1986)

Beurling's problem

Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$.

Find:

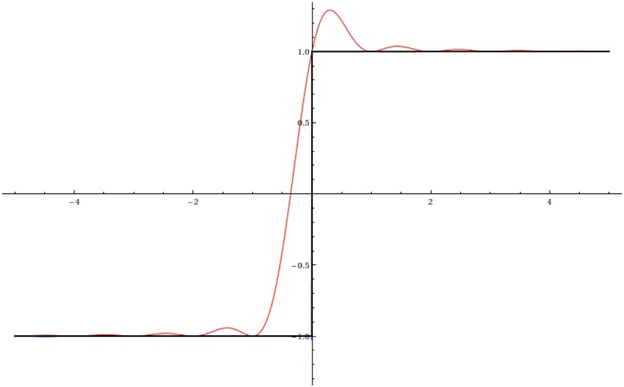
$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} (f(x) - \operatorname{sgn}(x)) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f\in\mathcal{E}}\int_{-\infty}^{\infty}(f(x)-\mathrm{sgn}(x))\mathrm{d}x=1.$$

$$F(z)=\left(\frac{\sin\pi z}{\pi}\right)^2\left\{\sum_{n=0}^{\infty}\frac{1}{(z-n)^2}-\sum_{n=-\infty}^{-1}\frac{1}{(z-n)^2}+\frac{2}{z}\right\}.$$

$$F(x) = \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} \frac{1}{(x-n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(x-n)^2} + \frac{2}{x} \right\}.$$



Beurling's problem

$\Rightarrow$

Hilbert's inequality (Montgomery-Vaughan)

Selberg's problem - 1974

Atle Selberg (1917 - 2007)
Medalha Fields (1950)

Selberg's problem

Let $a, b \in \mathbb{R}$ such that $b - a \in \mathbb{Z}$. Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \chi_{[a,b]}(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \left(f(x) - \chi_{[a,b]}(x)\right) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f\in\mathcal{E}} \int_{-\infty}^{\infty} \left(f(x) - \chi_{[a,b]}\right)\mathrm{d}x = 1.$$

The function is not unique.

For all $x \in \mathbb{R} \setminus \{a, b\}$ we have

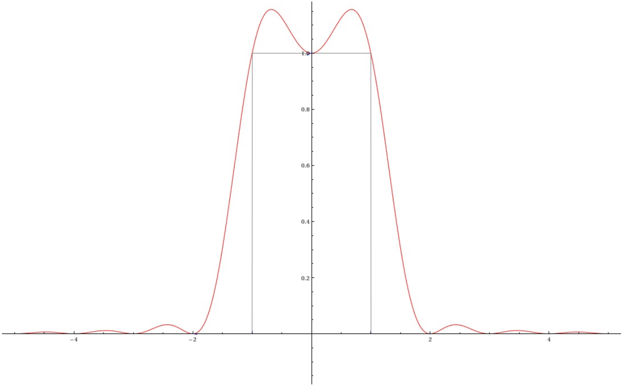
$$\chi_{[a,b]} = \frac{1}{2}\big(\operatorname{sgn}(b-x) + \operatorname{sgn}(x-a)\big).$$

Let F be the solution of the sgn-problem. An answer for the Selberg's problem can be:

$$G(z) = \frac{1}{2}\big(F(b-z) + F(z-a)\big).$$

When $a = -1, b = 1$ we have

$$G(z) = \frac{1}{2}\big(F(1-z) + F(z+1)\big).$$



Selberg's problem

$\Rightarrow$

Selberg - Sharp Form of the large sieve inequality

Extremal problem: Majorant

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $\mathcal{E}$ be the set of functions $M : \mathbb{C} \to \mathbb{C}$ such that:

1. M is a real entire function of exponential type 2π.
2. $M(x) \geq f(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{M \in \mathcal{E}} \int_{-\infty}^{\infty} \big(M(x) - f(x)\big)\mathrm{d}x.$$

Extremal problem: Minorant

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $\mathcal{E}$ be the set of functions $m : \mathbb{C} \to \mathbb{C}$ such that:

1. m is a real entire function of exponential type 2π.
2. $f(x) \geq m(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{m \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - m(x)\big)\mathrm{d}x.$$

Function	Answer
$\mathrm{sgn}(x)$	Beurling 1930's
$\chi_{[a,b]}(x)$	Selberg 1970's and Logan 1980's
$e^{-\lambda\|x\|}$, $\mathrm{sgn}(x)e^{-\lambda\|x\|}$	Graham - Vaaler 1981
Even functions ($\log\|x\|$)	Carneiro - Vaaler 2009
Even functions ($e^{-\lambda x^2}$) (Gaussian subordination)	Carneiro - Littmann - Vaaler 2010
Odd functions ($\mathrm{sgn}(x)e^{-\lambda x^2}$) (odd Gaussian subordination)	Carneiro - Vaaler 2011

Theorem (Hilbert)

Let $a_1, a_2, \cdots, a_n$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{m-n} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2.$$

Theorem (Hilbert)

Let $a_1, a_2, \cdots, a_n$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{m-n} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2.$$

1. *Hilbert proved the result with constant 2π.*

Theorem (Hilbert)

Let $a_1, a_2, \cdots, a_n$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{m-n} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2.$$

1. *Hilbert proved the result with constant 2π.*
2. *Schur proved the result with constant π (optimal constant).*

Theorem (Montgomery-Vaughan)

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be real numbers such that $|\lambda_m - \lambda_n| \geq \delta > 0$ when $m \neq n$. Let $a_1, a_2, \cdots, a_n$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq \pi \sum_{n=1}^{N} \frac{|a_n|^2}{\delta}.$$

Beurling's problem

Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \operatorname{sgn}(x)$ for all $x \in \mathbb{R}$.

Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \operatorname{sgn}(x)\big) \mathrm{d}x.$$

Beurling's problem

Let $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \mathrm{sgn}(x)$ for all $x \in \mathbb{R}$.

Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \mathrm{sgn}(x)\big)\mathrm{d}x.$$

Answer: 1

Proposition (Vaaler)

Let F be the solution of the Beurling's problem. Let

$$\psi(x) = F(x) - \operatorname{sgn}(x).$$

Então

1. $\psi(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. $\psi \in L^1(\mathbb{R})$ *and* $\widehat{\psi}(0) = 1$.
3. $\widehat{\psi}(t) = -\dfrac{1}{\pi i t}$ *for all* $|t| \geq 1$.

Proof of Montgomery-Vaughan inequality!