Class 12: Weighted Hilbert's inequality

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Theorem (Approximation formula)

Let $C > 1$ *and* $\sigma_0 > 0$. *Then, for* $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ *and* $|t| < \frac{2\pi x}{C}$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ *be analytic in the strip* $|\mathrm{Im}\, s| \leq \frac{1}{2} + \varepsilon$ *for some* $\varepsilon > 0$, *and assume that* $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ *for some* $\delta > 0$ *when* $|\mathrm{Re}\, s| \to \infty$. *Then*

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi} \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Example

1. Let $\chi_{[-1,1]}$ be the characteristic function of the interval $[-1, 1]$. Then

$$\widehat{\chi_{[-1,1]}}(t) = \frac{\sin 2\pi t}{\pi t}.$$

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2. The function $f(x) = \text{máx}\{1-|x|, 0\}$ has
$$\widehat{f}(t) = \left(\frac{\sin \pi t}{\pi t}\right)^2.$$

Example

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$$\widehat{\chi_{[-1,1]}}(t) = \frac{\sin 2\pi t}{\pi t}.$$

2. The function $f(x) = \max\{1-|x|, 0\}$ has

$$\widehat{f}(t) = \left(\frac{\sin \pi t}{\pi t}\right)^2.$$

3. The function $L(x) = \dfrac{\sin \pi x}{\pi x}$ satisfies

$$\widehat{L}(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t).$$

Entire functions of exponential type

Definition

We say that an entire function has exponential type $2\pi\delta$*, if for any* $\varepsilon > 0$ *there is* $C_\varepsilon > 0$ *such that*

$$|f(z)| \leq C_\varepsilon e^{(2\pi\delta+\varepsilon)|z|}$$

for all $z \in \mathbb{C}$*.*

Paley-Wiener theorem

Theorem

Let F be an entire function such that $F \in L^2(\mathbb{R})$. The following statements are equivalent:

1. *F has exponential type $2\pi\delta$.*
2. *$\widehat{F}$ has compact support in $[-\delta, \delta]$ a.e. ($\widehat{F}(t) = 0$ for $|t| \geq \delta$ a.e.)*

Example

1. $L(z) = \dfrac{\sin \pi z}{\pi z}$ has exponential type π and satisfies
$$\widehat{L}(t) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(t).$$
2. $G(z) = \left(\dfrac{\sin \pi z}{\pi z}\right)^2$ has exponential type 2π and satisfies
$$\widehat{G}(t) = \text{máx}\{1 - |t|, 0\}.$$

Interpolation formulas

Theorem (Shannon-Whittaker)

Let F be an entire function of exponential type π such that $F \in L^2(\mathbb{R})$. Então

$$F(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \frac{F(n)}{z - n},$$

where the series converges uniformly in compacts of $\mathbb{C}$.

Interpolation formulas

Theorem (Vaaler, 1985)

Let F be an entire function of exponential type 2π such that $F \in L^2(\mathbb{R})$. Então

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n\in\mathbb{Z}} \frac{F(n)}{(z-n)^2} + \sum_{n\in\mathbb{Z}} \frac{F'(n)}{(z-n)} \right\},$$

where the series converges uniformly in compacts of $\mathbb{C}$.

Beurling-Selberg problem

Delta problem

Let $\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$\delta(x) = \begin{cases} 1 & \textit{if} \quad x = 0 \\ 0 & \textit{if} \quad x \neq 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \delta(x)$ for all $x \in \mathbb{R}$.

Find:

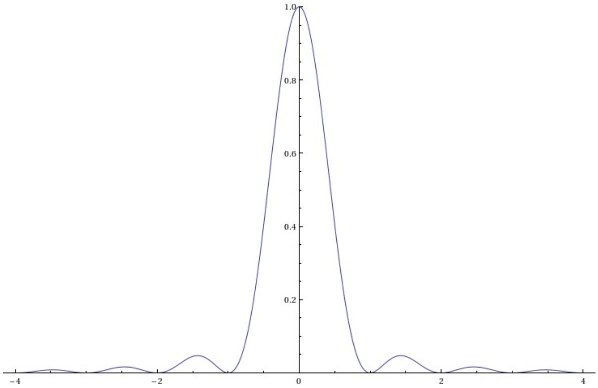
$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \delta(x)\big) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1.$$

$$F(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2 .$$

$$F(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2.$$



Delta problem

$\Rightarrow$

Pair correlation of the zeros of the Riemann zeta-function, multiplicity of the zeros of zeta.

Beurling's problem - 1930's

Arne Carl-August Beurling (1905 - 1986)

Beurling's problem

Let $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \mathrm{sgn}(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} (f(x) - \mathrm{sgn}(x)) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f\in\mathcal{E}} \int_{-\infty}^{\infty} (f(x) - \mathrm{sgn}(x))\mathrm{d}x = 1.$$

$$F(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} + \frac{2}{z} \right\}.$$

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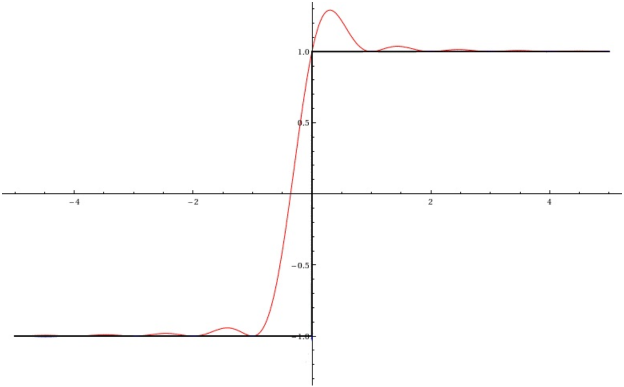


Beurling's problem

$$\Rightarrow$$

Hilbert's inequality (Montgomery-Vaughan)

Selberg's problem - 1974

Atle Selberg (1917 - 2007)
Medalha Fields (1950)

Selberg's problem

Let $a, b \in \mathbb{R}$ such that $b - a \in \mathbb{Z}$. Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \chi_{[a,b]}(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \left(f(x) - \chi_{[a,b]}(x)\right) \mathrm{d}x.$$

Answer of the problem

$$\inf_{f\in\mathcal{E}}\int_{-\infty}^{\infty}\left(f(x)-\chi_{[a,b]}\right)\mathrm{d}x=1.$$

The function is not unique.

For all $x \in \mathbb{R} \setminus \{a, b\}$ we have

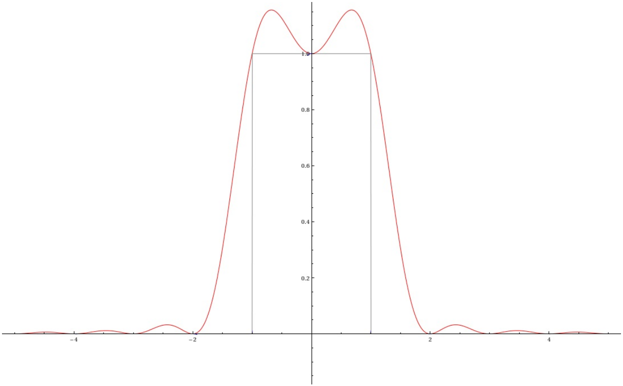
$$\chi_{[a,b]} = \frac{1}{2}\big(\operatorname{sgn}(b-x) + \operatorname{sgn}(x-a)\big).$$

Let F be the solution of the sgn-problem. An answer for the Selberg's problem can be:

$$G(z) = \frac{1}{2}\big(F(b-z) + F(z-a)\big).$$

When $a = -1, b = 1$ we have

$$G(z) = \frac{1}{2}\big(F(1-z) + F(z+1)\big).$$



Selberg's problem

$\Rightarrow$

Selberg - Sharp Form of the large sieve inequality

Extremal problem: Majorant

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $\mathcal{E}$ be the set of functions $M : \mathbb{C} \to \mathbb{C}$ such that:

1. M is a real entire function of exponential type 2π.
2. $M(x) \geq f(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{M \in \mathcal{E}} \int_{-\infty}^{\infty} \big(M(x) - f(x)\big)\mathrm{d}x.$$

Extremal problem: Minorant

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Let $\mathcal{E}$ be the set of functions $m : \mathbb{C} \to \mathbb{C}$ such that:

1. m is a real entire function of exponential type 2π.
2. $f(x) \geq m(x)$ for all $x \in \mathbb{R}$.

Find:

$$\inf_{m \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - m(x)\big)\,\mathrm{d}x.$$

Function	Answer
$\mathrm{sgn}(x)$	Beurling 1930's
$\chi_{[a,b]}(x)$	Selberg 1970's and Logan 1980's
$e^{-\lambda\lvert x\rvert}$, $\mathrm{sgn}(x)e^{-\lambda\lvert x\rvert}$	Graham - Vaaler 1981
Even functions ($\log\lvert x\rvert$)	Carneiro - Vaaler 2009
Even functions ($e^{-\lambda x^2}$) (Gaussian subordination)	Carneiro - Littmann - Vaaler 2010
Odd functions ($\mathrm{sgn}(x)e^{-\lambda x^2}$) (odd Gaussian subordination)	Carneiro - Vaaler 2011

Theorem (Hilbert)

Let $a_1, a_2, \cdots, a_N$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{m-n} \right| \leq \pi \sum_{n=1}^{N} |a_n|^2.$$

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1. *Hilbert proved the result with constant 2π.*

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1. *Hilbert proved the result with constant 2π.*
2. *Schur proved the result with constant π (optimal constant).*

Theorem (Montgomery-Vaughan)

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be real numbers such that $|\lambda_m - \lambda_n| \geq \delta > 0$ when $m \neq n$. Let $a_1, a_2, \cdots, a_N$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq \pi \sum_{n=1}^{N} \frac{|a_n|^2}{\delta}.$$

Beurling's problem

Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
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$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \operatorname{sgn}(x)\big) \mathrm{d}x$$

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$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq C \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}.$$

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1. Montgomery-Vaughan (1971) proved the result with constant $3\pi/2$.
2. Preissman (1983) proved the result with constant $4\pi/3$.

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Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be real numbers such that $|\lambda_m - \lambda_n| \geq \delta_n > 0$ when $m \neq n$. Let $a_1, a_2, \cdots, a_N$ be complex numbers. Then

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1. Montgomery-Vaughan (1971) proved the result with constant $3\pi/2$.
2. Preissman (1983) proved the result with constant $4\pi/3$.
3. Conjecture: π.

Variation of the Beurling's problem

Let $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
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3. f is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$.

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3. f is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$.

Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - \operatorname{sgn}(x)\big) \mathrm{d}x.$$

x_0^+-problem

Let $x_0^+ : \mathbb{R} \to \mathbb{R}$ defined by

$$x_0^+(x) = \begin{cases} 1 & si \quad x > 0 \\ \frac{1}{2} & si \quad x = 0 \\ 0 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq x_0^+(x)$ for all $x \in \mathbb{R}$.
3. f is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$.

Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \left(f(x) - x_0^+(x)\right) \mathrm{d}x.$$

Answer of the problem: Carneiro and Littmann

$$\inf_{f\in\mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - x_0^{+}(x)\big)\mathrm{d}x = 1.$$

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

Proposition

Let F be the function defined by:

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

Then:

1. *F is a real entire function of exponential type 2π.*
2. *$F(x) \geq x_0^+(x)$ for all $x \in \mathbb{R}$.*
3. *F is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$.*
4. *$F - x_0^+ \in L^1(\mathbb{R})$, and*

$$\int_{-\infty}^{\infty} (F(x) - x_0^+(x))\,\mathrm{d}x = A.$$

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We have

$$F(x) - F(0) = \int_0^x F'(s)\,\mathrm{d}s = \int_0^x -\frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

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Then, F is a real entire function of exponential type 2π.

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

We have

$$F(x) - F(0) = \int_0^x F'(s)\,\mathrm{d}s = \int_0^x -\frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

Then, F is a real entire function of exponential type 2π. Taking derivative, trivially we have that F is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$.

When $x < 0$, using the expression

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s,$$

we get trivially that $F(x) \geq 0$.

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we get trivially that $F(x) \geq 0$.
Now, assume that $x > 0$. Using the fact that

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we get trivially that $F(x) \geq 0$.
Now, assume that $x > 0$. Using the fact that

$$-\int_{-\infty}^{\infty} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s = 1,$$

it follows that

$$\begin{aligned} F(x) - 1 &= -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s + \int_{-\infty}^{\infty} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s \\ &= \int_{x}^{\infty} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s \geq 0. \end{aligned}$$

We write

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s = -\int_{-\infty}^{x} g(s)\,\mathrm{d}s.$$

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To prove that $F - x_0^+ \in L^1(\mathbb{R})$ we use the fact that for $x < 0$,

$$F(x) - x_0^+(x) = F(x) = -\int_{-\infty}^{x} g(s)\,\mathrm{d}s,$$

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and, for $x > 0$,

$$\begin{aligned} F(x) - x_0^+(x) = F(x) - 1 &= -\int_{-\infty}^{x} g(s)\,\mathrm{d}s + \int_{-\infty}^{\infty} g(s)\,\mathrm{d}s \\ &= \int_{x}^{\infty} g(s)\,\mathrm{d}s. \end{aligned}$$

Proposition

Let F be the solution of the previous problem. Let

$$\psi(x) = F(x) - x_0^+(x).$$

1. $\psi(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. ψ *is increasing in* $(-\infty, 0)$ *and decreasing in* $(0, \infty)$.
3. $\psi \in L^1(\mathbb{R})$.
4. $\widehat{\psi}(0) = A$.
5. $\widehat{\psi}(t) = -\frac{1}{2\pi i t}$, *for* $|t| \geq 1$.

Proposition

Let G be the solution of the modified sgn *problem. Let*

$$\Psi(x) = G(x) - \operatorname{sgn}(x).$$

1. $\Psi(x) \geq 0$ *for all* $x \in \mathbb{R}$.
2. Ψ *is increasing in* $(-\infty, 0)$ *and decreasing in* $(0, \infty)$.
3. $\Psi \in L^1(\mathbb{R})$.
4. $\widehat{\Psi}(0) = 2A$.
5. $\widehat{\Psi}(t) = -\frac{1}{\pi i t}$, *for* $|t| \geq 1$.

Proof of Montgomery-Vaughan inequality!

Assume that $\delta_1 \geq \delta_2 \geq ... \geq \delta_N > 0$ and $\Psi_{\delta_0} \equiv 0$.