Class 13: Mean values for $\zeta(s)$

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Theorem (Approximation formula)

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ and $|t| < \frac{2\pi x}{C}$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \le \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$

$$- \frac{1}{2\pi} \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right) \right)$$

$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Theorem (Montgomery-Vaughan)

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be real numbers such that $|\lambda_m - \lambda_n| \geq \delta_n > 0$ when $m \neq n$. Let $a_1, a_2, \cdots, a_N$ be complex numbers. Then

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq C \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}.$$

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4. Conjecture: π.

Variation of the Beurling's problem

Let $\mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & si \quad x > 0 \\ 0 & si \quad x = 0 \\ -1 & si \quad x < 0 \end{cases}$$

Let $\mathcal{E}$ be the set of functions $f : \mathbb{C} \to \mathbb{C}$ such that:

1. f is a real entire function of exponential type 2π.
2. $f(x) \geq \mathrm{sgn}(x)$ for all $x \in \mathbb{R}$.

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Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} (f(x) - \mathrm{sgn}(x)) \mathrm{d}x.$$

x_0^+-problem

Let $x_0^+ : \mathbb{R} \to \mathbb{R}$ defined by

$$x_0^+(x) = \begin{cases} 1 & si \quad x > 0 \\ \frac{1}{2} & si \quad x = 0 \\ 0 & si \quad x < 0 \end{cases}$$

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Find

$$\inf_{f \in \mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - x_0^+(x)\big) \mathrm{d}x.$$

Answer of the problem: Carneiro and Littmann

$$\inf_{f\in\mathcal{E}} \int_{-\infty}^{\infty} \big(f(x) - x_0^{+}(x)\big)\mathrm{d}x = 1.$$

$$F(x) = -\int_{-\infty}^{x} \frac{\sin^2(\pi s)}{\pi^2 s(s+1)^2}\,\mathrm{d}s.$$

Mean value of $\zeta(s)$: the second moment

Estimate: for $\sigma > 0$ and $T > 0$ large

$$\frac{1}{T}\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t$$

Let $\sigma > 1$. Then:

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T |\zeta(\sigma+it)|^2 \mathrm{d}t = \zeta(2\sigma).$$

For $\frac{1}{2} \leq \sigma \leq 1$, we want to study:

$$\int_1^T |\zeta(\sigma + it)|^2 \mathrm{d}t$$

It is time to use the things that we have learned until now!

Theorem (Approximation formula)

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

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$$\zeta(\sigma + it) = \sum_{n \leq 2T} \frac{1}{n^{\sigma+it}} - \frac{2^{1-(\sigma+it)} T^{1-(\sigma+it)}}{1 - (\sigma + it)} + O(T^{-\sigma}),$$

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$$\zeta(\sigma+it) = \sum_{n \le 2T} \frac{1}{n^{\sigma+it}} + O(T^{-\sigma}).$$

Therefore:

$$\int_T^{2T} |\zeta(\sigma+it)|^2 \mathrm{d}t$$

$$= \int_T^{2T} \left| \sum_{n\le 2T} \frac{1}{n^{\sigma+it}} \right|^2 \mathrm{d}t + O\left(T^{1-\sigma} \sum_{n\le 2T} \frac{1}{n^{\sigma}} \right) + O\big(T^{1-2\sigma}\big).$$

$$\int_T^{2T} \left| \sum_{n\le 2T} \frac{1}{n^{\sigma+it}} \right|^2 \mathrm{d}t = \sum_{m,n\le 2T} \frac{1}{m^\sigma n^\sigma} \int_T^{2T} \left(\frac{m}{n}\right)^{it} \mathrm{d}t$$

$$= \sum_{n\le 2T} \frac{1}{n^{2\sigma}}\, T + \sum_{\substack{m,n\le 2T \\ m\neq n}} \frac{1}{m^\sigma n^\sigma} \left(\frac{(m/n)^{2iT} - (m/n)^{iT}}{i\log(m/n)} \right)$$

$$= \sum_{n\le 2T} \frac{1}{n^{2\sigma}}\, T + \sum_{\substack{m,n\le 2T \\ m\neq n}} \left(\frac{m^{-\sigma+2iT} n^{-\sigma-2iT}}{i(\log m - \log n)} \right)$$

$$+ \sum_{\substack{m,n\le 2T \\ m\neq n}} \left(\frac{m^{-\sigma+iT} n^{-\sigma-iT}}{i(\log m - \log n)} \right).$$

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Montgomery-Vaughan inequality! 

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$$\left|\sum_{\substack{m,n\le 2T\\ m\ne n}} \frac{m^{-\sigma+iT}n^{-\sigma-iT}}{i(\log m-\log n)}\right| \le C\sum_{n\le 2T}\frac{|n^{-\sigma+iT}|^2}{\frac{1}{2n}}.$$

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$$\left|\sum_{\substack{m,n\le 2T\\ m\ne n}} \frac{m^{-\sigma+iT}n^{-\sigma-iT}}{i(\log m-\log n)}\right| \le 2C\sum_{n\le 2T}\frac{1}{n^{2\sigma-1}}.$$

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Therefore:

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Therefore, for $T \geq 2$ we have:

1. If $\sigma = \frac{1}{2}$:

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^2 \mathrm{d}t = T \log T + O(T).$$

2. If $\frac{1}{2} < \sigma < 1$:

$$\int_T^{2T} |\zeta(\sigma + it)|^2 \mathrm{d}t = \zeta(2\sigma) T + O\left(\frac{T^{2-2\sigma}}{1-\sigma}\right).$$

3. If $\sigma = 1$:

$$\int_T^{2T} |\zeta(1 + it)|^2 \mathrm{d}t = \zeta(2) T + O(\log T).$$

We conclude using a dyadic argument.

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5. **Conjecture:** $E(T) \ll T^{1/4+\varepsilon}$.

Conjecture

For $k > 0$

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} \mathrm{d}t \sim C_k T (\log T)^{k^2}.$$

1. Hardy and Littlewood (1918): $k = 1$.

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3. For the other values it is unknown.
4. Currently, there are some lower bounds and upper bounds, assuming RH and also without assuming RH (Heath-Brown, Radziwill, Soundararajan, Harper, Winston Heap).