Class 14: Lindelöf hypothesis

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Theorem (Approximation formula)

Let $C > 1$ and $\sigma_0 > 0$. Then, for $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ and $|t| < \frac{2\pi x}{C}$.

Lemma (Guinand-Weil explicit formula)

Let $h(s)$ be analytic in the strip $|\mathrm{Im}\, s| \le \frac{1}{2} + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\mathrm{Re}\, s| \to \infty$. Then

$$\sum_{\rho} h\left(\frac{\rho - \frac{1}{2}}{i}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \left\{ \mathrm{Re}\, \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) - \log \pi \right\} \mathrm{d}u$$
$$- \frac{1}{2\pi} \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{h}\left(\frac{\log n}{2\pi}\right) + \widehat{h}\left(\frac{-\log n}{2\pi}\right)\right)$$
$$+ h\left(\frac{1}{2i}\right) + h\left(-\frac{1}{2i}\right)$$

Theorem (Montgomery-Vaughan)

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ *be real numbers such that* $|\lambda_m - \lambda_n| \geq \delta_n > 0$ *when* $m \neq n$*. Let* $a_1, a_2, \cdots, a_N$ *be complex numbers. Then*

$$\left| \sum_{\substack{m,n=1 \\ n \neq m}}^{N} \frac{a_m \overline{a_n}}{\lambda_m - \lambda_n} \right| \leq C \sum_{n=1}^{N} \frac{|a_n|^2}{\delta_n}.$$

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4. Conjecture: π.

Therefore, for $T \geq 2$ we have:

1. If $\sigma = \frac{1}{2}$:

$$\int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^2 \mathrm{d}t = T \log T + O(T).$$

2. If $\frac{1}{2} < \sigma < 1$:

$$\int_T^{2T} |\zeta(\sigma + it)|^2 \mathrm{d}t = \zeta(2\sigma) T + O\left(\frac{T^{2-2\sigma}}{1-\sigma}\right).$$

3. If $\sigma = 1$:

$$\int_T^{2T} |\zeta(1 + it)|^2 \mathrm{d}t = \zeta(2) T + O(\log T).$$

For $T \geq 3$ we have:

$$\int_1^T |\zeta(\tfrac{1}{2} + it)|^2 \mathrm{d}t = T \log T - (1 + \log 2\pi - 2\gamma) T + O(E(T)).$$

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5. Conjecture: $E(T) \ll T^{1/4+\varepsilon}$.

Now, we want to bound $\zeta(s)$ in the critical strip!

For $\mathrm{Re}\, s > 1$ we have

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Then, for $\mathrm{Re}\, s \geq 1 + \delta > 1$, we have

$$|\zeta(s)| \leq C_\delta.$$

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Then, $s = \sigma + it$, with $\frac{1}{2} \leq \sigma \leq 1 + \delta$, $|t| \geq 2$ we have

$$\zeta(s) = O(|t|).$$

Therefore, we conclude that for $\sigma \geq \frac{1}{2}$:

$$\zeta(s) = O(|t|).$$

Recalling the functional equation:

$$\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \pi^{-(1-s)/2}\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right).$$

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Then, we write

$$\zeta(s) = \chi(s)\,\zeta(1-s),$$

where

$$\chi(s) = \frac{\pi^{-(1-2s)/2}\,\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}.$$

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using your favorite Stirling's formula: for a fixed $\delta > 0$ and $-\pi + \delta < \arg(s) < \pi - \delta$, show that

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O(|s|^{-1}),$$

as $|s| \to \infty$.

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$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O(|s|^{-1}),$$

as $|s| \to \infty$. Then, we get for any fixed strip $\alpha \leq \sigma \leq \beta$, as $t \to \infty$:

$$\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma + it - \frac{1}{2}} e^{i(t + \frac{\pi}{4})}\left(1 + O\left(\frac{1}{t}\right)\right).$$

Then, for $\alpha \leq \sigma \leq \beta$, as $t \to \infty$:

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and

$$\zeta(s) = O(|t|^{3/2+\delta}), \quad \text{for } \sigma \geq -\delta.$$

Therefore, for any semiplane $\sigma \geq \sigma_0$ we have

$$|\zeta(s)| = O(|t|^k),$$

for some k depending on σ_0. This implies that $\zeta(s)$ is a function of finite order in the sense of the theory of Dirichlet series.

For any σ we define $\mu(\sigma)$ as the infimum of the values ξ such that

$$\zeta(\sigma + it) = O\big(|t|^{\xi}\big).$$

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Then:

$$\mu\left(\frac{1}{2}\right) \leq 1.$$

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Theorem

Let $C > 1$ *and* $\sigma_0 > 0$. *Then, for* $x \geq 1$,

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O_{\sigma_0, C}(x^{-\sigma}),$$

uniformly in $0 < \sigma_0 \leq \sigma \leq 1$ *and* $|t| < \frac{2\pi x}{C}$.

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we have

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq t} \frac{1}{n^{\frac{1}{2}+it}} - \frac{t^{\frac{1}{2}-it}}{\frac{1}{2} - it} + O(t^{-\frac{1}{2}}).$$

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This implies that

$$\mu\left(\frac{1}{2}\right)\le\frac{1}{2}.$$

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Let $f : \Omega \to \mathbb{C}$ be a holomorphic function such that Ω contains the strip $\sigma_1 \leq \mathrm{Re}\, s \leq \sigma_2$. Suppose that $f(\sigma + it) = O\big(e^{\varepsilon|t|}\big)$ in the strip $\sigma_1 \leq \mathrm{Re}\, s \leq \sigma_2$, for every $\varepsilon > 0$. Suppose that $f(\sigma_1 + it) = O\big(|t|^{k_1}\big)$ and $f(\sigma_2 + it) = O\big(|t|^{k_2}\big)$. Then, we have

$$f(\sigma + it) = O\big(|t|^{k(\sigma)}\big),$$

uniformly for $\sigma_1 \leq \sigma \leq \sigma_2$, where $k(\sigma)$ is the linear function of σ which takes the values k_1 and k_2 for $\sigma = \sigma_1$ and $\sigma = \sigma_2$ respectively.

We apply this result for $\zeta(\sigma + it)$. Let $\sigma_1 \leq \sigma \leq \sigma_2$. We have that $\zeta(\sigma_1 + it) = O(|t|^{\mu(\sigma_1)+\varepsilon})$ and $\zeta(\sigma_2 + it) = O(|t|^{\mu(\sigma_2)+\varepsilon})$. Then, for $\sigma_1 \leq \sigma \leq \sigma_2$:

$$\zeta(\sigma + it) = O(|t|^{k(\sigma)}),$$

where $k(\sigma) = \dfrac{(\sigma_2 - \sigma)(\mu(\sigma_1) + \varepsilon) + (\sigma - \sigma_1)(\mu(\sigma_2) + \varepsilon)}{\sigma_2 - \sigma_1}$.

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where $k(\sigma) = \dfrac{(\sigma_2 - \sigma)(\mu(\sigma_1) + \varepsilon) + (\sigma - \sigma_1)(\mu(\sigma_2) + \varepsilon)}{\sigma_2 - \sigma_1}$.

Therefore, for $\sigma_1 \leq \sigma \leq \sigma_2$:

$$\mu(\sigma) \leq \frac{(\sigma_2 - \sigma)\mu(\sigma_1) + (\sigma - \sigma_1)\mu(\sigma_2)}{\sigma_2 - \sigma_1}.$$

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Therefore, for $\sigma_1 \leq \sigma \leq \sigma_2$:

$$\mu(\sigma) \leq \frac{(\sigma_2 - \sigma)\mu(\sigma_1) + (\sigma - \sigma_1)\mu(\sigma_2)}{\sigma_2 - \sigma_1}.$$

We conclude that $\mu(\sigma)$ is a convex function.

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4. $\mu(\sigma) = 0$, *for* $\sigma \geq 1$.
5. $\mu(\sigma) = \frac{1}{2} - \sigma$, *for* $\sigma \leq 0$.
6. μ *is a decreasing function.*

In particular, using the fact that $\mu(0) = \frac{1}{2}$ and $\mu(1) = 0$, it follows for $0 < \sigma < 1$:

$$\mu(\sigma) \leq \frac{1}{2} - \frac{\sigma}{2}.$$

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This is called: **Convexity bound**

Lindelöf hypothesis-1908

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$$\left|\zeta\left(\frac{1}{2}+it\right)\right| = O\left(|t|^{\varepsilon}\right).$$

$\mu(1/2) \leq$	$\mu(1/2) \leq$	Author	
1/4	0.25	Lindelöf (1908)	Convexity bound
1/6	0.1667	Hardy, Littlewood & ?	
163/988	0.1650	Walfisz (1924)	
27/164	0.1647	Titchmarsh (1932)	
229/1392	0.164512	Phillips (1933)	
	0.164511	Rankin (1955)	
19/116	0.1638	Titchmarsh (1942)	
15/92	0.1631	Min (1949)	
6/37	0.16217	Haneke (1962)	
173/1067	0.16214	Kolesnik (1973)	
35/216	0.16204	Kolesnik (1982)	
139/858	0.16201	Kolesnik (1985)	
32/205	0.1561	Huxley (2002, 2005)	
53/342	0.1550	Bourgain (2017)	
13/84	0.1548	Bourgain (2017)	

Riemann hypothesis implies Lindelöf hypothesis

The next class with Bondarenko!